

NOTES ON DIVERGENCES AND DIMENSIONAL TRANSMUTATION IN YANG–MILLS THEORY

L. D. Faddeev*

We discuss the specificity of charge renormalization in Yang–Mills theory. We show that the values of the running coupling constant in dimensional regularization and in momentum truncation coincide. Dimensional transmutation is interpreted as replacing the dimensionless coupling constant with a dimensional invariant of the renormalization group equation.

Keywords: dimensional transmutation, renormalization group equations, dimensional regularization, momentum truncation

Since its creation in the beginning of the 1930s, quantum field theory has been haunted by ultraviolet divergences. Renormalization theory has led to phenomenal advances in quantum electrodynamics, but it has given no satisfaction to its creators. The rebirth of interest in field theory in the form of Yang–Mills theory has posed the question of divergences anew. It has gradually become clear that divergences do not discredit the theory but on the contrary play a positive role as an effective means of violating conformal invariance in the classical theory. A correct interpretation of dimensional transmutation (this term was introduced in [1]) must solve the mass problem for the quantum Yang–Mills theory in the four-dimensional space–time.

In this paper, I present my interpretation of this notion. Most of my notes have been known to some extent, but the placement of accents may prove useful.

To begin, I consider an elementary example in nonrelativistic quantum mechanics, where dimensional transmutation is readily explicable and has a clear mathematical interpretation. I then consider Yang–Mills theory in the background field formalism, where charge renormalization is performed particularly simply and leads to the appearance of the running coupling constant $\alpha(\mu)$. I compare regularizations based on introducing the truncation momentum Λ and on changing $4 \rightarrow 4 - \epsilon$ in the space dimension and show that the values of $\alpha(\mu)$ for them coincide. The dimensional parameter m appears as an invariant of the renormalization group equation. In conclusion, I state a rather speculative hypothesis on how the parameter m can enter the physical parameters in Yang–Mills theory.

With great pleasure, I dedicate this paper to the jubilee of Yurii Viktorovich Novozhilov, whose lectures on quantum field theory I attended in my student days more than 50 years ago.

1. An elementary example

We consider the Schrödinger operator for a nonrelativistic particle located in the two-dimensional space and interacting with a center concentrated at the origin,

$$H = -\Delta + \epsilon\delta^{(2)}(x) = H_0 + V.$$

*St. Petersburg Department of the Steklov Institute of Mathematics, RAS, St. Petersburg, Russia, e-mail: faddeev@pdmi.ras.ru.

Translated from *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 148, No. 1, pp. 133–142, July, 2006. Original article submitted March 2, 2006.

The main object of scattering theory is the resolvent $R(z) = (H - zI)^{-1}$ with the structure

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z),$$

where $R_0(z)$ is given by the formula $R_0(z) = (H_0 - zI)^{-1}$ and the T -matrix $T(z)$ satisfies the equation [2]

$$T(z) = V - VR_0(z)T(z). \quad (1)$$

In the momentum space, the potential V is an integral operator with kernel,

$$V(p, p') = \frac{1}{(2\pi)^2} \int e^{i(p-p')x} V(x) d^2x.$$

In our example, we have $V(p, p') = \epsilon$. It follows straightforwardly from Eq. (1) that $T(p, p'; z)$ is independent of both p and p' , $T(p, p'; z) = t(z)$, and the integral equation itself reduces to the algebraic relation

$$t(z) = \epsilon - \epsilon \int \frac{d^2p}{p^2 - z} t(z) \quad (2)$$

involving a divergent integral. We regularize it by introducing a truncation momentum Λ . We have

$$\int_{|p| \leq \Lambda} \frac{d^2p}{p^2 - z} = \pi \int_0^{\Lambda^2} \frac{dx}{x - z} = \pi \log \frac{\Lambda^2}{-z},$$

where the principal branch of the logarithm regarded as a function of z is considered, as a result of which the right-hand side of the relation is defined on the z plane with a cut along the positive half-axis corresponding to the continuous spectrum.

We rewrite regularized relation (2) in the form

$$\frac{1}{t(z)} = \frac{1}{\epsilon} + \pi \log \frac{\Lambda^2}{-z} = \frac{1}{\epsilon} + \pi \log \frac{\Lambda^2}{m^2} + \pi \log \frac{m^2}{-z}$$

and, following the renormalization theory paradigm stemming from Landau and Wilson, assume that the coupling constant ϵ depends on Λ . We set

$$\frac{1}{\epsilon(\Lambda)} + \pi \log \frac{\Lambda^2}{m^2} = 0. \quad (3)$$

It is clear that this can hold only for $\epsilon(\Lambda) < 0$, which corresponds to attraction. We rewrite (3) in the form

$$m = \Lambda \exp\left(\frac{1}{2\pi\epsilon}\right)$$

and assume that $\epsilon(\Lambda) \rightarrow 0$ as $\Lambda \rightarrow \infty$, in view of which the value of m remains finite. The new parameter m enters the formula for the T -matrix,

$$t(z) = \left(\pi \log \frac{m^2}{-z}\right)^{-1},$$

in which no trace of the seeding coupling constant ϵ has remained. As can be seen, $t(z)$ has a simple pole at $z = -m^2$.

It can be easily verified that the operator $R(z)$ constructed using $T(z)$ is in fact the resolvent of a self-adjoint operator with a continuous spectrum on the positive half-axis and a discrete eigenvalue at the point $-m^2$. As was explained long ago in [3], this operator is a self-adjoint extension of a symmetric operator determined by the Laplace operator $-\Delta$ on functions vanishing at the origin. In [3], we considered the “realistic” case of the three-dimensional space instead of the “artificial” two-dimensional case, which is in fact much more instructive.

Indeed, formula (3) gives an example of a phenomenon that was called dimensional transmutation in [1], which is connected with violation of scale invariance. We elucidate this in greater detail. Under the change of variable $x \rightarrow \lambda x$, the Laplace operator $-\Delta$ and the function $\delta^2(x)$ undergo similar transformations. In other words, they have the same dimensionality $[L]^{-2}$, as a result of which the coupling constant ϵ is dimensionless. At the same time, the new parameter m in the resolvent has a dimensionality, $m = [L]^{-1}$, and it fixes the scale in the operator H . It can be said that the scale covariance of the operator H is violated. This violation is caused by the divergence that appears in formula (2), and it indicates the senselessness of this formal expression at a finite value of ϵ . Only if we pass to the limit assuming that ϵ is a function of the truncation momentum Λ and letting it tend to zero do we obtain a meaningful result.

We are thus convinced that divergences can play a positive role leading to interesting possibilities. This idea has not yet become widespread, but it has already been shared by many specialists, among whom we mention Jackiw [4], 't Hooft [5], and Wilczek [6]. Its realization seems particularly attractive in Yang–Mills theory, where it must lead to describing massive excitations.

2. Background field formalism for Yang–Mills theory

Charge renormalization in Yang–Mills theory is realized particularly simply in the background field formalism. The main object of this formalism [7] is the effective action $W(\Phi^{\text{ph}})$ equal to the sum of 1PI vacuum diagrams obtained by integrating the functional $e^{iS(\Phi)}$ after the change of variable

$$\Phi = \Phi^{\text{ph}} + \varphi.$$

The background field Φ^{ph} must satisfy some physical boundary conditions and an equation of motion that uniquely determines the field from these conditions. This equation coincides with the classical one only in the zeroth order in \hbar and involves quantum corrections.

The background field formalism is more suitable for Yang–Mills theory than the commonly accepted formalism that uses external sources and Green’s functions and is hardly compatible with gauge invariance. For our purposes, an important feature is that the only renormalization in the background field formalism is charge renormalization [8].

Because the main objects of the formalism are well known, we confine ourselves to only recalling the formulas needed to elucidate the notation. The Yang–Mills field is described by the connection 1-form

$$A = A_\mu^a t_a dx^\mu$$

with the curvature

$$F = dA + A \wedge A,$$

where t^a are the generators of the corresponding compact charge group G with the normalization $\text{tr}(t^a t^b) = 2\delta^{ab}$ and tr is the Killing form.

We consider only the four-dimensional space–time with a plane metric and use the Euclidean formulation for definiteness. The classical action is given by the formula

$$S = \frac{1}{4g^2} \int \text{tr} F \wedge F^*,$$

where g^2 is the coupling constant serving as the only parameter characterizing the theory. In the system of units with $\hbar = 1$ and $c = 1$, this parameter has no dimensionality because the forms A and F are dimensionless.

The shift

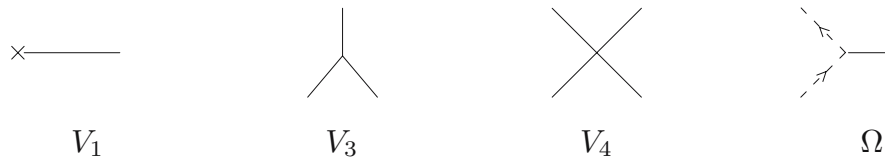
$$A = A^{\text{ph}} + ga$$

in the functional integral leads to a diagram technique with the following elements:

the propagators



and the vertices



for the vector fields and the ghosts. The vertices V_1 , V_3 , V_4 , and Ω have the respective orders $1/g$, g , g^2 , and g . All these elements depend on the background field A^{ph} . The coefficient function at the vertex V_1 coincides with the classical equation of motion. The quantum equation of motion for the background field can be represented as

$$\times\text{---} + \text{---}\text{---} = 0,$$

where the second term involves only 1PI diagrams.

Integrating over the variable a and the ghosts results in an expansion for the effective action,

$$W(A^{\text{ph}}) = \frac{1}{\alpha}W_{-1} + W_0 + \alpha W_1 + \alpha^2 W_2 + \dots$$

Here and hereafter, for convenience, we use the notation $\alpha = g^2/(4\pi)$, and W_{-1} is the classical action,

$$W_{-1} = \frac{1}{16\pi} \int \text{tr } F \wedge F^*.$$

Furthermore, we have $W_0 = \log \det M_0 - (1/2) \log \det M_1$, where M_0 and M_1 are differential operators whose Green's functions are the corresponding propagators G_0 and G_1 , and the W_k are given by the sum of 1PI vacuum diagrams with $k+1$ loops, for instance,

$$W_1 + \text{---}\text{---} + \text{---}\text{---} + \text{---}\text{---}.$$

The presented expression involves divergences, and we now discuss them.

3. Regularization and renormalization

Divergences in the background field formalism have been discussed by many authors (see, e.g., [9] and [10]). The main result is that the divergent contributions to W_0 and W_1 have the structure

$$W_k = C_k \widetilde{W}_{-1} + \text{finite part},$$

where C_k are divergent constants. The structure of divergences for W_k , $k > 1$, is more complicated, but they all involve divergent terms proportional to W_{-1} . As a result, the effective action acquires the form

$$W(A^{\text{ph}}) = \left(\frac{1}{\alpha} + C_0 + C_1 \alpha + C_2 \alpha^2 + \dots \right) W_{-1}(A^{\text{ph}}) + \dots,$$

and it is natural to call the sum in the parentheses a renormalized charge,

$$\frac{1}{\alpha_r} = \left(\frac{1}{\alpha} + C_0 + C_1 \alpha + C_2 \alpha^2 + \dots \right).$$

The renormalizability means that the divergences in multiloop diagrams are also combined to form type $(\alpha_r)^m$ factors, as a result of which the functionals \widetilde{W}_k in the effective action expressed in terms of α_r ,

$$W(A^{\text{ph}}) = \frac{1}{\alpha_r} W_{-1} + \widetilde{W}_0 + \alpha_r \widetilde{W}_1 + \dots,$$

contain no divergences.

After the regularization, the constants C_k depend on the corresponding parameter, namely, on the truncation momentum Λ or on the dimension defect ϵ in the case of dimensional regularization. We consider these regularizations consecutively.

Dimensional considerations show that in the first case, the C_k are polynomials in $\log(\Lambda/\mu)$, where Λ is the truncation momentum and μ is an auxiliary parameter needed to undimensionalize Λ . It appears as the upper integration limit in the regularization of an integral of the form

$$\int_0^{\Lambda} \frac{ds}{s} \sim \log \frac{\Lambda}{\mu}$$

and has the meaning of infrared truncation. It is clear that the renormalized charge depends on μ . Some general considerations (see below) show that

$$\frac{1}{\alpha_r(\mu)} = \frac{1}{\alpha(\Lambda)} + c_0 \log \frac{\Lambda}{\mu} + c_1 \log \frac{\Lambda}{\mu} \alpha(\Lambda) + \left(c_{21} \log \frac{\Lambda}{\mu} + c_{22} \log^2 \frac{\Lambda}{\mu} \right) \alpha^2(\Lambda) + \dots, \quad (4)$$

where

$$c_0 = -\frac{11}{3} \frac{K_2(G)}{2\pi}, \quad c_1 = -\frac{17}{3} \left(\frac{K_2(G)}{2\pi} \right)^2.$$

Here, $K_2(G)$ is the normalization of the Casimir operator in the adjoint representation. We have $K_2(G) = N$ for the group $SU(N)$. The most important property of this formula is that c_0 and c_1 are negative. This precisely gives hope that $\alpha_r(\mu)$ is defined in the limit as $\Lambda \rightarrow \infty$, $\alpha(\Lambda) \rightarrow 0$.

Contrary to the example in Sec. 1, we cannot set $\alpha_r(\mu) = \infty$ here, because the higher loops involve divergences that are not proportional to W_{-1} . Setting $\Lambda = \mu$ in (4), we see that $\alpha_r(\mu) = \alpha(\mu)$ and that

the nonrenormalized and renormalized coupling constants represent the values of the same function $\alpha(x)$ for $x = \Lambda$ and for $x = \mu$.

We must confess that a satisfactory calculation of even the two-loop contribution in this regularization has not yet been elaborated. Therefore, the assertion that formula (4) holds is still hypothetical, and we state it proceeding from the compatibility with the renormalization group as explained in the next section.

In dimensional regularization, which was used in [9] and [10] mentioned above, the expansion

$$W(A^{\text{ph}}) = \left(\frac{1}{\alpha} + \frac{b_0}{\epsilon} + \frac{b_1}{\epsilon} \alpha + \frac{b_{21}}{\epsilon} \alpha^2 + \frac{b_{22}}{\epsilon^2} \alpha^2 \right) W_{-1} + \dots$$

was obtained, where

$$b_0 = c_0, \quad b_1 = \frac{c_1}{2}, \quad b_{21} = \frac{c_{21}}{3}.$$

At first glance, it seems that the renormalized charge does not contain the parameter μ . But a more detailed consideration shows how this dependence can appear (cf. [9]). In the space of dimension $D = 4 - \epsilon$, the nonrenormalized charge has a dimensionality, $\alpha = \alpha_0(\epsilon)\mu^\epsilon$, where α_0 is dimensionless, as a result of which we obtain the expansion

$$\frac{1}{\tilde{\alpha}(\mu)} = \frac{1}{\alpha(\epsilon)} + \frac{b_0}{\epsilon} \mu^{-\epsilon} + \frac{b_1}{\epsilon} \alpha(\epsilon) \mu^{-2\epsilon} + \alpha^2(\epsilon) \mu^{-3\epsilon} \left(\frac{b_{21}}{\epsilon} + \frac{b_{22}}{\epsilon^2} \right) + \dots \quad (5)$$

for the renormalized charge $\tilde{\alpha}(\mu)$, and the dependence on μ does not disappear as $\epsilon \rightarrow 0$. As is shown in the next section, the functions $\tilde{\alpha}(\mu)$ and $\alpha(\mu)$ coincide.

4. Renormalization group and dimensional transmutation

An important consequence of renormalization theory is the Gell-Mann–Low differential equation according to which the running coupling constant $\alpha(\Lambda)$ satisfies the differential equation

$$\Lambda \frac{d\alpha(\Lambda)}{d\Lambda} = \beta(\alpha(\Lambda)), \quad (6)$$

where the right-hand side is not explicitly dependent on Λ . Equation (6) means that by varying the truncation momentum, we can fit the coupling constant $\alpha(\Lambda)$ such that the physical results are independent of Λ .

Differentiating with respect to Λ , we conclude that formula (4) is compatible with Eq. (6) if we assume that $\beta(\alpha)$ has an expansion of the form

$$\beta(\alpha) = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \beta_3 \alpha^4 + \dots$$

and that

$$\beta_1 = c_0, \quad \beta_2 = c_1, \quad \beta_3 = c_{21}, \quad c_{22} = -\frac{1}{2} \beta_1 \beta_2.$$

Of course, the renormalized charge $\alpha(\mu)$ satisfies the same equation as $\alpha(\Lambda)$. This can be expressed by the relation

$$\int_{\alpha(\mu)}^{\alpha(\Lambda)} \frac{dx}{\beta(x)} = \log \frac{\Lambda}{\mu}.$$

We now show that $\tilde{\alpha}(\mu)$ coincides with $\alpha(\mu)$. Differentiating (5), we verify that $\tilde{\alpha}(\mu)$ satisfies the equation

$$\mu \frac{d\tilde{\alpha}(\mu)}{d\mu} = \beta(\tilde{\alpha}(\mu))$$

if $b_0 = \beta_1$, $b_1 = \beta_2/2$, and $b_{22} = -\beta_1\beta_2/6$. Precisely these values of the coefficients b_0 , b_1 , and b_{22} were obtained in [11], which can be seen after a simple recalculation and correction of a misprint. Both regularizations thus lead to the same expression for the effective action in the form of an expansion in the running coupling constant $\alpha(\mu)$. This expansion involves no divergences.

We now discuss dimensional transmutation. The parameter μ in the running coupling constant can be arbitrary. The physics of the phenomenon must be independent of the choice of μ . The dimensional parameter appears as a trajectory invariant for the renormalization group. We elucidate this following Polyakov [12]. We must assume something about the global behavior of the β -function. We assume that it does not change sign and that the integral $\int dx/\beta(x)$ converges in the neighborhood of the point at infinity. We introduce a function $\theta(\alpha)$ by the relation

$$\theta(\alpha) = \int_{\alpha}^{\infty} \frac{dx}{\beta(x)}.$$

This function has negative values, it behaves as $1/(\beta_1\alpha)$ at small values of α , and tends monotonically to zero as $\alpha \rightarrow \infty$. The relation

$$\int_{\alpha(\mu_1)}^{\alpha(\mu_2)} \frac{dx}{\beta(x)} = \log \frac{\mu_2}{\mu_1}$$

can be rewritten as

$$\theta(\alpha(\mu_2)) - \theta(\alpha(\mu_1)) = \log \frac{\mu_2}{m} - \log \frac{\mu_1}{m}$$

or as

$$\theta(\alpha(\mu)) = -\log \frac{\mu}{m}, \tag{7}$$

where m plays the role of a separation constant. Rewriting (7) in the form

$$m = \mu e^{\theta(\alpha(\mu))},$$

we conclude that m is an invariant of the renormalization group. This is precisely the dimensional parameter in terms of which the effective action must be expressed. Polyakov calls m^{-1} the correlation length [12]. We note that the negativity of the function θ imposes a constraint on μ , $\mu \geq m$.

Rewriting (7) differently, we obtain an expression for the running coupling constant $\alpha(\mu)$,

$$\alpha(\mu) = \theta^{-1}\left(\log \frac{m}{\mu}\right),$$

where $\theta^{-1}(x)$ is the inverse function of $\theta(x)$. It maps the negative half-axis onto the positive half-axis, and we have

$$\theta^{-1}(x) \sim \frac{1}{\beta_1 x} \quad \text{as } x \rightarrow -\infty,$$

$$\theta^{-1}(x) \rightarrow \infty \quad \text{as } x \rightarrow -0.$$

In the example in Sec. 1, the parameter m^2 played the role of the coupling energy. Naturally, its ratio to the parameter z in the resolvent was dimensionless. In Yang–Mills theory, this parameter must also enter a combination with a dimensional physical parameter. The role of this dimensional parameter will most probably be played by the vacuum energy density, which in turn is generated by a nontrivial condensate.

We present a hypothetical pattern for this phenomenon. Together with Niemi [13], we discussed a change of variables for Yang–Mills $SU(2)$ fields and the corresponding stringlike excitations (see [14] for a generalization to the case $SU(3)$). For our program to succeed, we need a condensate for the field

$$\rho^2(x) = (A_\mu^1)^2 + (A_\mu^2)^2.$$

We set $\langle \rho^2 \rangle = H^2$. Many works have recently been devoted to discussing such a condensate of dimensionality $[L]^{-2}$ (see, e.g., [15] and the references therein).

The hypothesis is that the energy density is determined by the minimum of a function of the type presented by Savvidy [16],

$$E(H) = -\frac{H^4}{\theta^{-1}(\log(H/m))}.$$

Here, the denominator involves $\alpha(m^2/H)$. The function $E(H)$ is defined for $0 < H < m$, and it is negative at $H = 0$ and $H = m$. The physical value of H is defined as the position of the minimum for $E(H)$.

Of course, the above argument is still purely speculative, and only further investigation can reveal its potentialities.

At this point, we end the notes on dimensional transmutation in Yang–Mills theory.

Acknowledgments. When working at the text of this paper, the author had stimulating discussions with his young collaborators T. Bolokhov and P. Mnev and also with I. Aref’eva, F. Wilczek, A. Vladimirov, R. Jackiw, D. D’yakonov, D. Kazakov, A. Niemi, A. Slavnov, and G. ’t Hooft. The author expresses his gratitude to all of them.

This work was supported in part by the Russian Foundation for Basic Research (Grant No. 05-01-00922), the CRDF (Grant No. RUM-1-2622-ST-04), and the program “Mathematical Problems of Nonlinear Dynamics” of the Russian Academy of Sciences.

REFERENCES

1. S. Coleman and E. Weinberg, *Phys. Rev. D*, **7**, 1888 (1973).
2. S. P. Merkur’ev and L. D. Faddeev, *Quantum Scattering Theory for Several Particle Systems* [in Russian], Nauka, Moscow (1985); English transl.: L. D. Faddeev and S. P. Merkuriev, (Math. Phys. Appl. Math., Vol. 11), Kluwer, Dordrecht (1993).
3. F. A. Berezin and L. D. Faddeev, *Sov. Math. Dokl.*, **2**, 372 (1961).
4. R. Jackiw, “What good are quantum field theory infinities,” in: *Mathematical Physics 2000* (International Congress, London, GB, 2000, A. Fokas, A. Grigoryan, T. Kibble, and B. Zegarlinski, eds.), Imperial College Press, London (2000), p. 101.
5. G. ’t Hooft, *Phys. Stat. Solidi* (b), **237**, 13 (2003); hep-th/0208054 (2002).
6. F. Wilczek, “Four big questions with pretty good answers,” in: *Fundamental Physics: Heisenberg and Beyond* (Werner Heisenberg Centennial Symposium “Developments in Modern Physics,” G. W. Buschhorn and J. Wess, eds.), Springer, Berlin (2004), p. 79; hep-ph/0201222 (2002).
7. G. ’t. Hooft, *Nucl. Phys. B*, **62**, 444 (1973); B. S. DeWitt, *Dynamic Theory of Groups and Fields*, Gordon and Breach, New York (1965).
8. L. D. Faddeev, “Quantum theory of gauge fields,” in: *Vector Mesons and Electromagnetic Interactions* [in Russian] (Proc. Intl. Seminar, Dubna, 23–26 September 1969, A. M. Baldin, ed.), Joint Inst. Nucl. Res., Dubna (1969), p. 13.
9. L. F. Abbot, *Nucl. Phys. B*, **185**, 189 (1982).
10. I. Jack and H. Osborn, *Nucl. Phys. B*, **207**, 474 (1982).
11. J. P. Bornsen and A. E. M. van de Ven, *Nucl. Phys. B*, **657**, 257 (2003).

12. A. M. Polyakov, *Gauge Fields and Strings* (Contemporary Concepts in Physics, Vol. 3), Harwood, Chur, Switzerland (1987).
13. L. D. Faddeev and A. Niemi, *Phys. Lett. B*, **525**, 195 (2002).
14. L. D. Faddeev and T. A. Bolokhov, *Theor. Math. Phys.*, **139**, 679 (2004).
15. F. V. Gubarev, L. Stodolsky, and V. I. Zakharov, *Phys. Rev. Lett.*, **86**, 2220 (2001); K. I. Kondo, *Phys. Lett. B*, **514**, 335 (2001); H. Verschelde, K. Knecht, K. Van Acoleyen, and M. Vanderkelen, *Phys. Lett. B*, **516**, 307 (2001); hep-th/0105018 (2001).
16. G. K. Savvidy, *Phys. Lett. B*, **71**, 133 (1977).