A lower estimate for the modified Steiner functional

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We prove inequality (1) for the modified Steiner functional A(M), which extends the notion of the integral of mean curvature for convex surfaces. We also establish an expression for A(M) in terms of an integral over all hyperplanes intersecting the polyhedral surface M.

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1. Introduction

In the articles [1], [4], [5] the authors suggest a new version of string theory, which can be considered as a natural extension of the Feynman–Kac integral over paths to an integral over surfaces. Both amplitudes coinside in the case, when the surface degenarates into a single partical world line.

The string has been conjectured to describe a wide variety of physical phenomena, including strong interaction, the three dimensional Ising model, and unified models incorporating gravity. The Feynman integral for the string is just the partition function for the randomly fluctuating surfaces, and in this statistical approach the surface is associated with a connected polyhedral surface embedded in euclidean space.

To prove the convergence of the partition function for this new string, the authors of [4], [5] require a lower estimate for the action $A(M)$ on which the theory is based. The purpose of the present note is to prove the inequality

$$A(M) > 2\pi\Delta, \tag{1}$$

where $A(M)$ is the modified Steiner functional as introduced in [1], [4], [5] and Δ is the diameter of the polyhedral surface M in R^d. We also establish an expression for A(M) in terms of an integral over all hyperplanes intersecting the polyhedral surface M.

2.Proof of the inequality

We recall the definition of $A(M)$.

Definition. Let M be an embedded connected closed polyhedral surface in euclidean space R^d $(d \geq 3)$. Let $\mathcal{F}_1(M)$ be the set of edges of M. For $e \in \mathcal{F}_1(M)$ we denote by $L(e)$ the length of e and by $\alpha(e)$, where $0 < \alpha(e) < \pi$, the angle between the two faces of M incident with e. Then the modified Steiner functional is defined by

$$A(M) := \sum_{e\in\mathcal{F}_1(M)} L(e)[\pi - \alpha(e)].$$

Theorem. *If Δ denotes the diameter of M, then*

$$A(M) > 2\pi\Delta. \tag{2}$$

Proof. We first consider a simple closed polygon P in R^d. For a vertex v of P, we denote by $\alpha(v)$, where $0 < \alpha(v) < \pi$, the angle between the two edges of P incident with v. The (absolute) total curvature of P is defined by

$$\kappa(P) := \sum_{v}[\pi - \alpha(v)],$$

where the sum extends over the vertices of P. It is known that

$$\kappa(P) \geq 2\pi \tag{3}$$

(Fenchel's inequality for polygons; see, e.g., [2]).

In the proof of inequality (1) we shall use some integral geometry, in particular the space $\mathcal{E}_{d-1}^d$ of hyperplanes in R^d with its (suitably normalized) rigid motion invariant measure μ_{d-1}; see, e.g., [6]. According to [6], (1.9), the measure μ_{d-1} can be represented as follows. For a nonnegative measurable function f on $\mathcal{E}_{d-1}^d$ we have

$$\int\limits_{\mathcal{E}_{d-1}^d} f d\mu_{d-1} = \int\limits_{S^{d-1}} \int\limits_{-\infty}^{\infty} f(H_{u,\tau}) d\tau d\sigma(u).$$

Here $S^{d-1} := \{u \in \mathsf{R}^d : \|u\| = 1\}$ is the unit sphere of R^d,

$$H_{u,\tau} := \{x \in \mathsf{R}^d : \langle x, u \rangle = \tau\}, \quad u \in S^{d-1}, \ \tau \in \mathsf{R},$$

is a general hyperplane with unit normal vector u, and σ is the spherical Lebesgue measure on S^{d-1}, normalized to total measure 1. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in R^d.

The hyperplane $H \in \mathcal{E}_{d-1}^d$ is said to intersect the polyhedral surface M *in general position* if $H \cap M \neq \emptyset$ and H does not contain a vertex of M. In that case, the intersection of H with an edge of M is either empty or a point, and the intersection of H with a face of M is either empty or a segment. It follows that the intersection $H \cap M$ is the union of finitely many simple closed polygons $P_1(H), \ldots, P_k(H)$, and from inequality (3) (applied in H instead of R^d) we have

$$\kappa(H \cap M) := \kappa(P_1(H)) + \ldots + \kappa(P_k(H)) \geq 2\pi.$$

It follows that

$$I := \int\limits_{\mathcal{E}_{d-1}^d} \kappa(H \cap M) d\mu_{d-1}(H) \geq 2\pi \mu_{d-1}(\{H \in \mathcal{E}_{d-1}^d : H \cap M \neq \emptyset\}), \tag{4}$$

since the set of all hyperplanes intersecting M, but not in general position, has μ_{d-1}- measure zero. Let S be a segment connecting two points of M with maximal distance, so that the length of S is equal to the diameter Δ of M. Let s be a unit vector parallel to S. Then

$$\mu_{d-1}(\{H \in \mathcal{E}_{d-1}^d : H \cap M \neq \emptyset\}) = \int\limits_{S^{d-1}} \int\limits_{-\infty}^{\infty} \mathbf{1}_{\{H_{u,\tau} \cap M \neq \emptyset\}} d\tau d\sigma(u)$$

$$> \int\limits_{S^{d-1}} \int\limits_{-\infty}^{\infty} \mathbf{1}_{\{H_{u,\tau} \cap S \neq \emptyset\}} d\tau d\sigma(u) = \int\limits_{S^{d-1}} \Delta|\langle u, s \rangle| d\sigma(u)$$

$$= c_1 \Delta.$$

Here $\mathbf{1}_X$ denotes the indicator function of X. By $c_1, \ldots, c_6$ we denote constants depending only on the dimension d. We have proved that

$$I > c_2 \Delta. \tag{5}$$

On the other hand, if the hyperplane H intersects M in general position, we can write

$$\kappa(H \cap M) = \sum_{e \in \mathcal{F}_1(M)} [\pi - \beta(M, e, H)], \tag{6}$$

where $\beta(M, e, H)$ is defined as follows. If H meets the edge e (and hence the relative interior of e) and if F_1, F_2 are the two faces of M incident with e, then $\beta(M, e, H) \in (0, \pi)$ is the angle between the segments $H \cap F_1$ and $H \cap F_2$ at the point $H \cap e$. If H does not meet e, we put $\beta(M, e, H) = \pi$. We can now write

$$I = \sum_{e \in \mathcal{F}_1(M)} \int_{\mathcal{E}_{d-1}^d} [\pi - \beta(M, e, H)] d\mu_{d-1}(H). \tag{7}$$

Let $e \in \mathcal{F}_1(M)$ be a fixed edge. We have

$$\int_{\mathcal{E}_{d-1}^d} [\pi - \beta(M, e, H)] d\mu_{d-1}(H)$$

$$= \int_{S^{d-1}} \int_{-\infty}^{\infty} [\pi - \beta(M, e, H_{u,\tau})] d\tau d\sigma(u)$$

$$= \int_{S^{d-1}} \left[\pi - \beta(M, e, H_{u,\langle x, u \rangle}) \right] L(e) |\langle u, w(e) \rangle| d\sigma(u), \tag{8}$$

where x is some fixed point in the relative interior of the edge e and $w(e)$ denotes a unit vector parallel to the edge e.

Let F_1, F_2 be the two faces of M incident with e and let $\alpha(e) \in (0, \pi)$ be the angle between F_1 and F_2, as defined initially. We assert that

$$\int_{S^{d-1}} \left[\pi - \beta(M, e, H_{u,\langle x, u \rangle}) \right] |\langle u, w(e) \rangle| d\sigma(u) = c_5 [\pi - \alpha(e)]. \tag{9}$$

For the proof we may assume, without loss of generality, that x is the origin of R^d. The integral in (9) can be written in the form

$$J := \int_{\mathcal{L}_{d-1}^d} f(H) d\nu_{d-1}(H), \tag{10}$$

where $\mathcal{L}_{d-1}^d$ denotes the space of $(d-1)$-dimensional linear subspaces of R^d and ν_{d-1} is its normalized invariant measure; the function f is defined by

$$f(H) = [\pi - \beta(M, e, H)] |\langle u_H, w(e) \rangle|,$$

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where u_H is a unit normal vector of H. The edge e and the two adjacent faces F_1, F_2 of M lie in a 3-dimensional linear subspace A of R^d. For ν_{d-1}- almost all $H \in \mathcal{L}^d_{d-1}$, the intersection $A \cap H$ is a 2-dimensional linear subspace. In that case, the angle $\beta(M, e, H)$ depends only on M and this subspace, so that we can write $\beta(M, e, H) = \beta(M, e, A \cap H)$. Moreover,

$$|\langle u_H, w(e)\rangle| = |\langle u_H, u_{A\cap H}\rangle \langle u_{A\cap H}, w(e)\rangle|,$$

where $u_{A\cap H}$ is a unit normal vector of $A \cap H$ in A.

Using a general formula of integral geometry, one can write the integral (10) in the form

$$\int_{\mathcal{L}^d_{d-1}} f(H)d\nu_{d-1}(H) = c_3 \int_{\mathcal{L}^A_2} \int_{\mathcal{L}^L_{d-1}} f(H)[H, A]^2 d\nu^L_{d-1}(H)d\nu^A_2(L).$$

Here $\mathcal{L}^A_2$ denotes the space of 2-dimensional linear subspaces of A and ν^A_2 is the normalized invariant measure on this space. For fixed $L \in \mathcal{L}^A_2$, $\mathcal{L}^L_{d-1}$ denotes the space of hyperplanes containing L, and ν^L_{d-1} is the invariant measure on this space. $[H, A]$ is a certain function depending only on the relative position of H and A; it is invariant under simultaneous rotations of H and A. The identity above is equivalent to a special case of formula (14.40) in Santaló [3], but written in the style of [6]. Applying this to our present situation, we obtain

$$
\begin{aligned}
J &= c_3 \int_{\mathcal{L}^A_2} \int_{\mathcal{L}^L_{d-1}} [\pi - \beta(M, e, H)] \, |\langle u_H, u_{A\cap H}\rangle \langle u_{A\cap H}, w(e)\rangle| \, [H, A]^2 d\nu^L_{d-1}(H)d\nu^A_2(L) \\
&= c_3 \int_{\mathcal{L}^A_2} [\pi - \beta(M, e, L)] \, |\langle u_L, w(e)\rangle| \int_{\mathcal{L}^L_{d-1}} [H, A]^2 \, |\langle u_H, u_L\rangle| \, d\nu^L_{d-1}(H)d\nu^A_2(L) \\
&= c_4 \int_{\mathcal{L}^A_2} [\pi - \beta(M, e, L)] \, |\langle u_L, w(e)\rangle| \, d\nu^A_2(L).
\end{aligned}
$$

The final integral is a mean value over 2-dimensional linear subspaces in a 3-dimensional euclidean space. Its value can be obtained from the more general Theorem 3.2.1 in [6]. In this way we arrive at

$$J = c_5[\pi - \alpha(e)],$$

which proves (9).

Taking (4), (7), (8), (9) together, we deduce that

$$A(M) > c_6 \Delta. \tag{11}$$

In order to find the optimal constant c_6 for which (11) holds generally, we consider the boundary M_ϵ of a triangular prism with height Δ and base a regular triangle with edge length ϵ. For $\epsilon \to 0$, the diameter of M_ϵ tends to Δ and $A(M_\epsilon)$ tends to $2\pi\Delta$. It follows that $c_6 \leq 2\pi$. On the other hand, from

the way inequality (11) was obtained it is easy to see that $A(M) > A(M_\epsilon)$, if $\epsilon > 0$ is sufficiently small. Thus $c_6 = 2\pi$ is the optimal constant.

3. Concluding Remark

We want to stress that the represantaton (7) is very useful for studing more complicated models [1], [4], [5] and various phenomena.

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