Spikes in quantum Regge calculus

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Abstract. For most measures in two-dimensional quantum Regge calculus proposed in the literature we show that the average values of link lengths l, $\langle l^n \rangle$, do not exist for sufficiently high powers of n. In particular, this is also true for the nonlocal DeWitt-like measure introduced by Regge and Lund. Thus the concept of length has no natural definition in this formalism and a generic manifold degenerates into spikes. This might explain the failure of quantum Regge calculus to reproduce the continuum results of two-dimensional quantum gravity. It points to severe problems for the Regge approach in higher dimensions.

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1. Introduction

In the path integral formulation of quantum gravity we are instructed to integrate over all gauge invariant classes of metrics on a given manifold $\mathcal{M}$:

$$Z(\Lambda, G) = \int \mathcal{D}[g_{\mu\nu}] \mathrm{e}^{-S_{\mathrm{EH}}[g_{\mu\nu}]}, \tag{1}$$

where $[g_{\mu\nu}]$ denotes an equivalence class of metrics, i.e. the metrics related by a diffeomorphism $\mathcal{M} \mapsto \mathcal{M}$, and $S_{\mathrm{EH}}[g_{\mu\nu}]$ denotes the Einstein–Hilbert action:

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \int_{\mathcal{M}} \sqrt{g} \left(\Lambda - \frac{1}{16\pi G} R \right). \tag{2}$$

We have written the functional integral in Euclidean spacetime. In two-dimensional (2d) spacetime this integral is well defined. In higher dimensions special care has to be taken, either by adding higher derivative terms to the Einstein–Hilbert action or by performing special analytic continuations of the modes which in higher dimensions are responsible for the unboundedness of the Einstein–Hilbert action. In this paper we will mainly discuss 2d gravity, but as will be clear the simple underlying reason for the problems we encounter might persist in higher dimensions.

It is well know how to perform the functional integration (1). One introduces on the space of all metrics of a given manifold $\mathcal{M}$ the norm

$$\|\delta g_{\mu\nu}\|^2 = \int_{\mathcal{M}} \mathrm{d}^D x \, \sqrt{g} \delta g_{\mu\nu}(x) G^{\mu\nu\alpha\beta} \delta g_{\alpha\beta}(x), \tag{3}$$

where $G^{\mu\nu\alpha\beta}$ is the DeWitt metric given by

$$G^{\mu\nu\alpha\beta} = \tfrac{1}{2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + C g^{\mu\nu}g^{\alpha\beta}). \tag{4}$$

This metric can be derived from symmetry considerations and from the requirement of diffeomorphism invariance. The constant C determines the signature of the metric. Let D denote the dimension of spacetime. For $C > -2/D$, the metric $G^{\mu\nu\alpha\beta}$ is positive definite, and for $C < -2/D$, it has negative eigenvalues. In canonical quantum gravity C equals -2. Using the DeWitt metric, the functional measure on the space of all metrics can then be determined to be

$$\mathcal{D}g_{\mu\nu} = \prod_x \sqrt{\det(\sqrt{g}G^{\mu\nu\alpha\beta})} \prod_{\mu\leqslant\nu} \mathrm{d}g_{\mu\nu}(x)$$

$$\sim \prod_x \sqrt{1 + \frac{DC}{2}} g(x)^{\frac{1}{8}(D+1)(D-4)} \prod_{\mu\leqslant\nu} \mathrm{d}g_{\mu\nu}(x). \tag{5}$$

This is the unique ultralocal diffeomorphism-invariant functional integration measure for $g_{\mu\nu}$. We get the measure $\mathcal{D}[g_{\mu\nu}]$ in (1) by an appropriate gauge fixing such that we divide (5) by the volume of the diffeomorphism group.

This program has been carried out explicitly for 2d gravity by Knizhnik *et al*, David, and Distler and Kawai [1]. One finds

$$Z(\Lambda, G) \sim \Lambda^{\frac{5h-5}{2}} \mathrm{e}^{\frac{1-h}{2G}}, \tag{6}$$

where h denotes the number of handles of $\mathcal{M}$. The continuum derivation is usually based on Ward identities or a self-consistent bootstrap ansatz, although attempts of a derivation from first principle exist. However, *dynamical triangulations* (DT) represents a regularization of the path integral, where an explicit cut-off (the link length ε of each equilateral triangle) is introduced [3]. Each triangulation is viewed as representing an equivalence class of metrics since the triangulation uniquely defines the distances in the manifolds via the length assignment of the links (ε for all links) and the assumption of a flat metric in the interior of each triangle. The functional integral (1) is then replaced by the summation over all triangulations of a given topology, and the class of such metrics can be viewed as a grid in the class of all metrics. In the limit $\varepsilon \to 0$ the summation over this grid reproduces the continuum result (6).

An alternative and conceptually quite nice approach was studied in [4]. In these papers the continuum functional integral over all metrics is replaced by the continuum functional integral over so-called piecewise linear metrics, i.e. metrics which represent geometries which are flat except in a finite number of points where curvature is located. If one restricts the number of singularities to be V, say, one can repeat the usual continuum calculations except that the final functional integration over the Liouville field becomes a finite dimensional $(3V - 6)$-dimensional integral (if the manifold has spherical topology). It is believed that this integral converges (in some suitable sense) to the continuum value (6) for $V \to \infty$, although this has strictly speaking not been proven. The final, $(3V - 6)$-dimensional integral requires some explicit regularization and it has not yet been possible to perform this integration. However, since the measure used is the restriction of full continuum measure to the class of piecewise linear metrics, and since this class of metrics includes in particular all metrics used in DT it *should* give the correct answer.

In the rest of this article we will discuss a formulation of 2d quantum gravity, which we denote as *quantum Regge calculus* (QRC) and which has been used extensively as a regularization of quantum gravity (for some reviews and extensive references see [5]).

Classical Regge calculus is a coordinate independent discretization of general relativity. By choosing a fixed (suitable) triangulation and by viewing the link lengths as dynamical variables one can approximate certain aspects of smooth manifolds. Each choice of link lengths consistent with all triangle inequalities creates a metric assignment to the manifold

if we view the triangulation as flat in the interior of the simplexes. Not all such assignments correspond to inequivalent metrics, as is clear by considering triangulations of 2d flat space. If we divide out this additional invariance we obtain for a given triangulation with L links an L-dimensional subspace of the infinite dimensional space of equivalence classes of metrics. This counting is in agreement with the one above, since the relation between the number of links L and the number of vertices V where the curvature is located, is given by $L = 3V - 6$ in the case of spherical topology. Quantum Regge calculus is defined by performing the functional integral (1) on this finite dimensional space and then taking the limit $L \rightarrow \infty$. It is the hope that this (seemingly well defined) procedure, when applied to the calculation of expectation values of observables, will produce results which converge in some suitable way to the continuum values of the observables when $L \rightarrow \infty$.

At this point an important difference between 2d QRC and 2d DT emerges: the first procedure is a *discretization* of the continuum theory, while the latter in addition provides a *regularization*. In 2d QRC no cut-off related to geometry is needed [6], while the DT-formalism explicitly introduces a lattice cut-off ε. One can of course choose to introduce such a cut-off in 2d QRC too, since it is needed in higher-dimensional QRC [7]. However, it will make no difference for the arguments we present, and since ε should be taken to zero at the end of the calculation, we have chosen to work directly with $\varepsilon = 0$ for 2d QRC.

Numerical simulations have questioned if QRC agrees with continuum predictions [8].

We will show that QRC is unlikely to be useful in quantum gravity. More precisely, we show that it does not reproduce the continuum theory of quantum gravity associated with (6). It is difficult to say something rigorous about four-dimensional (4d) quantum gravity, since we have no well defined continuum theory with which we can compare. But the problems encountered in 2d QRC are likely to remain in higher-dimensional QRC.

There is presently no unique choice of measure to be used in QRC, in analogy with the DeWitt measure (5) in the continuum. *A priori* this should not be viewed as a problem. Also in the case of DT there is no unique choice, but *universality* in the sense introduced in the theory of critical phenomena, should ensure that any reasonable change of measure is irrelevant in the continuum limit. This has been verified in the context of DT by numerous numerical simulations. In the next section we discuss the QRC measures proposed in the literature. In section 3 we analyse the functional integrals which result from these measures and show that none of the measures will allow us to reproduce the continuum 2d quantum gravity theory. Section 4 contains our conclusions.

2. Regge integration measures

2.1. DeWitt-like measure

We can repeat the construction (3) of the norm of metric deformations under the restriction that the deformations are constrained to represent the metric deformations allowed by QRC. For alternative discussions, starting from the canonical approach to gravity, we refer to [5] where the corresponding metric is denoted as the *Lund–Regge metric*, referring to an unpublished work of Lund and Regge. Since our derivation is valid in any dimension and has as its starting point the functional formalism, we find it more appropriate to call the measure *DeWitt-like*.

First note that any D-simplex, in a D-dimensional piecewise linear manifold M, can be covered with charts (U, ϕ), where

$$U = \{\xi \in R_+^D | \xi_1 + \cdots + \xi_D < 1\} \tag{7}$$

and $\phi : U \to M$ is given by

$$\phi(\xi) = \xi_1 y_1 + \cdots + \xi_D y_D + (1 - \xi_1 - \cdots - \xi_D) y_{D+1}, \tag{8}$$

where $y_1, \ldots, y_{D+1}$ are the vertices of the D-simplex. In this way M is equipped with a manifold structure. On this manifold we use a 'canonical' metric, which on a D-simplex with chart (U, ϕ) is given by

$$g_{\mu\nu}(\xi) = \frac{\partial\phi\partial\phi}{\partial\xi^\mu\partial\xi^\nu} \tag{9}$$

This metric is compatible with the manifold structure and has the advantage that it can be expressed solely in terms of the link lengths of the piecewise linear manifold. In two dimensions (9) has the following form on a triangle:

$$g_{\mu\nu} = \begin{pmatrix} x_1 & \frac{1}{2}(x_1 + x_2 - x_3) \\ \frac{1}{2}(x_1 + x_2 - x_3) & x_2 \end{pmatrix}, \tag{10}$$

where x_i equals l_i^2, and l_i are the link lengths of the triangle. We assume here and everywhere below that the link lengths l_i satisfy the triangle inequalities.

The area A of the triangle can be expressed as

$$A = \int d^2\xi \sqrt{g(\xi)} = \tfrac{1}{2}\sqrt{g} = \tfrac{1}{2}(x_1 x_2 - \tfrac{1}{4}(x_1 + x_2 - x_3)^2)^{1/2}. \tag{11}$$

The fluctuation of the metric given in terms of the δx_i's is

$$\delta g_{\mu\nu} = \begin{pmatrix} \delta x_1 & \frac{1}{2}(\delta x_1 + \delta x_2 - \delta x_3) \\ \frac{1}{2}(\delta x_1 + \delta x_2 - \delta x_3) & \delta x_2 \end{pmatrix}. \tag{12}$$

For a single triangle, the line element (3) can now be computed:

$$
\begin{aligned}
\|\delta g_{\mu\nu}\|^2 &= \int d^2\xi \sqrt{g(\xi)} \delta g_{\mu\nu}(\xi) G^{\mu\nu\alpha\beta} \delta g_{\alpha\beta}(\xi) \\
&= \frac{A}{2}(2\delta g_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} + C\delta g_{\mu\nu} g^{\mu\nu} g^{\alpha\beta} \delta g_{\alpha\beta}) \\
&= -2A \det \delta g_{\mu\nu} \det g^{\mu\nu} \qquad (C = -2).
\end{aligned}
\tag{13}
$$

In the last line we have chosen the canonical value -2 for C, which greatly simplifies the resulting expressions. From (10) and (12) we obtain

$$
\begin{aligned}
\|\delta g_{\mu\nu}\|^2 &= \frac{1}{16A}((\delta x_1)^2 + (\delta x_2)^2 + (\delta x_3)^2 - 2\delta x_1 \delta x_2 - 2\delta x_1 \delta x_3 - 2\delta x_2 \delta x_3) \\
&= [\delta x_1, \delta x_2, \delta x_3]\frac{1}{16A}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}.
\end{aligned}
\tag{14}
$$

For a general 2d triangulation with L links the line element is given by the sum of the line elements (13) over all triangles:

$$
\begin{aligned}
\|\delta g_{\mu\nu}\|^2 &= \sum_T \int d^2\xi \sqrt{g^T(\xi)} \delta g_{\mu\nu}^T(\xi)(G^T)^{\mu\nu\alpha\beta} \delta g_{\alpha\beta}^T(\xi) \\
&= [\delta x_1, \ldots, \delta x_L]\mathbf{M}\begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_L \end{bmatrix}.
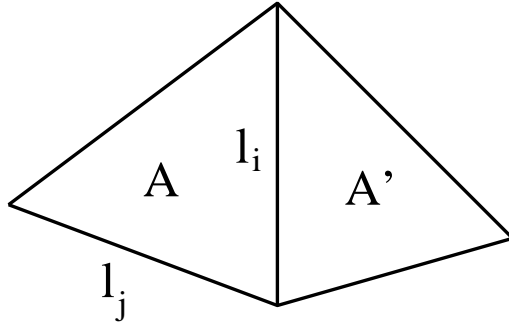\end{aligned}
\tag{15}
$$

Figure 1. Figure for explanation of the general structure of the matrix **M**.

M is an $L \times L$ matrix, which has the following structure: for a closed surface each link appears in two triangles in the sum (15). With the notation depicted in figure 1 we thus have the diagonal entry

$$M_{ii} = \frac{1}{A} + \frac{1}{A'}, \tag{16}$$

and the off-diagonal entry

$$M_{ij} = M_{ji} = -\frac{1}{A}. \tag{17}$$

Off-diagonal entries M_{ij} equal zero if l_i and l_j are not sides of the same triangle. It follows that each row and column of **M** has four nonvanishing off-diagonal entries for a closed surface. Thus the matrix **M** is a weighted adjacency matrix for the ϕ^4-graph which is constructed by connecting the midpoints of neighbouring links in the triangulation, with the weights given by (16) and (17).

Let T denote the number of triangles. From each row a factor $(AA')^{-1}$ can be factorized. Since each triangle has three sides, a factor $\prod_{k=1}^{T} A_k^{-3}$ can be factorized from the determinant of **M**:

$$\det \mathbf{M} = \prod_{k=1}^{T} A_k^{-3} \begin{vmatrix} A_1 + A_2 & -A_2 & -A_2 & -A_1 & -A_1 & 0 & \dots \\ & & \dots & & & \end{vmatrix}$$

$$=: P(A_1, \dots, A_T) \prod_{k=1}^{T} A_k^{-3}, \tag{18}$$

where $P(A_1, \dots, A_T)$ is a polynomial in the areas of the triangles of the surface. P vanishes whenever the area of two adjacent triangles vanish, for example when one link goes to zero. This means that P is a highly nonlocal function, because each monomial of P contains at least half of the areas of the surface.

From (15) we get that the DeWitt-like integration measure for QRC is given by the square root of the determinant of **M**:

$$d\mu(l_1, \dots, l_L) = \text{constant} \times \frac{\sqrt{P(A_1, \dots, A_T)}}{\prod_{k=1}^{T} A_k^{3/2}} \prod_{j=1}^{L} l_j \, dl_j \, \delta(\triangle), \tag{19}$$

where $\delta(\triangle)$ is a shorthand notation for the triangle inequalities satisfied by the links l_i.

We have not been able to obtain a closed form of P, but P can be determined in a number of special cases, as will be discussed below. In the next section we will only need

the fact that $P(A_i)$ is a function of the areas of the triangles and does not depend explicitly of the link length.

The continuum measure (5) in two dimensions is

$$[\mathcal{D}g_{\mu\nu}] = \text{constant} \times \prod_x g(x)^{-3/4} \prod_{\mu \leqslant \nu} \mathrm{d}g_{\mu\nu}(x). \tag{20}$$

Using $g(x) \sim A^2$, we see that the powers of the 'local' areas in the discretized measure coincide with the continuum measure. On the other hand the discretized measure is nonlocal, whereas the continuum measure is local.

Below we will analyse certain properties of the measure (19) and the analysis can be performed without further complications for the following generalization:

$$\mathrm{d}\mu(l_1, \ldots, l_L) = \text{constant} \times \frac{\sqrt{P(A_1, \ldots, A_T)}}{\prod_{k=1}^{T} A_k^{\beta}} \prod_{j=1}^{L} l_j \, \mathrm{d}l_j \, \delta(\triangle), \tag{21}$$

which appears when the measure (13) is multiplied by the 'local' area factor $A_i^{1-\beta}$. This generalization highlights that QRC does not fix the powers of A_i by any obvious principle, since the areas are reparametrization invariant objects which have no simple local continuum interpretation.

2.2. Other Regge measures

It is clear that the measure (19) is not suited for numerical simulations since it is highly nonlocal. In addition it should be clear that this measure is not forced upon us by the requirement of reparametrization invariance and ultralocality in the same way as the continuum DeWitt measure. It is not even the DeWitt measure restricted to the class of piecewise linear metrics. This measure was constructed in [4], as mentioned in the introduction. It is simply a 'translation' of the DeWitt measure to the rather special class of piecewise linear manifolds used in QRC. Thus it is natural to ask if there are other measures which are local and which are equally good. The most common local measures which have been used have the form (see [9] and references therein)

$$\mathrm{d}\mu(l_1, \ldots, l_L) = \prod_{j=1}^{L} \frac{\mathrm{d}l_j}{l_j^{\alpha}} \delta(\triangle) \tag{22}$$

and

$$\mathrm{d}\mu(l_1, \ldots, l_L) = \frac{\prod_{j=1}^{L} l_j \, \mathrm{d}l_j}{\prod_{k=1}^{T} A_k^{\beta}} \delta(\triangle). \tag{23}$$

The last measure is similar to the DeWitt measure except that the nonlocal term is missing.

One argument for such choices comes from the discretized Regge measure in one dimension. A one-dimensional (1d) Regge-manifold consists of L straight lines of length l_i ($x_i = l_i^2$), $i = 1, \ldots, L$ (see figure 2). The 'canonical metric' is thus $g_{\mu\nu}^i = [x_i]$. Using this, we can immediately write down the norm:

$$\|\delta g_{\mu\nu}\|^2 = \sum_{i=1}^{L} \int \mathrm{d}\xi \, \sqrt{g^i(\xi)} \delta g^i(\xi) G^i \delta g^i(\xi) = \tfrac{1}{2}(2 + C) \sum_i x_i^{-3/2} (\delta x_i)^2. \tag{24}$$

The measure is thus given by

$$\mathrm{d}\mu(l_1, \ldots, l_L) = \text{constant} \times \prod_{i=1}^{L} x_i^{-3/4} \, \mathrm{d}x_i = \text{constant} \times \prod_{i=1}^{L} l_i^{-1/2} \, \mathrm{d}l_i. \tag{25}$$
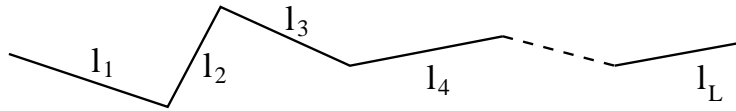
Figure 2. A 1d Regge-manifold.

In this case we have a a general factorization of $(1 + C/2D)$ as in the continuum and the measure is local†. However, this is not true in higher dimensions, as shown above. One of the purposes of the derivation of the DeWitt-like measure in $D = 2$ is to show that one cannot use (24) as a motivation for (22) and (23) in higher dimensions. Nevertheless, from the point of view of universality all measures discussed above should be equally good, provided they lead to the correct continuum limit.

2.3. The DeWitt-like measure for special geometries

In this subsection we derive the DeWitt-like measure for a number of simple 2d geometries.

2.3.1. Two triangles glued together. For the case of two triangles of area A glued together along the links, the total number of links L is 3, and the matrix $\mathbf{M}$ assumes the following form:

$$\mathbf{M} = \frac{1}{8A} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}. \tag{26}$$

The measure $d\mu(l_1, l_2, l_3)$ is then

$$d\mu(l_1, l_2, l_3) = (\det \mathbf{M})^{1/2} \prod_{j=1}^{3} l_j \, dl_j \, \delta(\triangle) = \text{constant} \times \frac{1}{A^{3/2}} \prod_{j=1}^{3} l_j \, dl_j \, \delta(\triangle). \tag{27}$$

2.3.2. Tetrahedron. For the tetrahedron, the number L of links is 6, and the number T of triangles is 4, see figure 3 for a parametrization. With this parametrization, the matrix $\mathbf{M}$ assumes the following form:

$$\mathbf{M} = \frac{1}{16} \begin{bmatrix} \frac{1}{A_1} + \frac{1}{A_3} & -\frac{1}{A_1} & -\frac{1}{A_3} & -\frac{1}{A_1} & 0 & -\frac{1}{A_3} \\ -\frac{1}{A_1} & \frac{1}{A_1} + \frac{1}{A_2} & -\frac{1}{A_2} & -\frac{1}{A_1} & -\frac{1}{A_2} & 0 \\ -\frac{1}{A_3} & -\frac{1}{A_2} & \frac{1}{A_2} + \frac{1}{A_3} & 0 & -\frac{1}{A_2} & -\frac{1}{A_3} \\ -\frac{1}{A_1} & -\frac{1}{A_1} & 0 & \frac{1}{A_1} + \frac{1}{A_4} & -\frac{1}{A_4} & -\frac{1}{A_4} \\ 0 & -\frac{1}{A_2} & -\frac{1}{A_2} & -\frac{1}{A_4} & \frac{1}{A_2} + \frac{1}{A_4} & -\frac{1}{A_4} \\ -\frac{1}{A_3} & 0 & -\frac{1}{A_3} & -\frac{1}{A_4} & -\frac{1}{A_4} & \frac{1}{A_3} + \frac{1}{A_4} \end{bmatrix}$$

The determinant of this matrix can be computed, and one obtains the result

$$\det \mathbf{M} = \text{constant} \times \frac{(\sum_{i=1}^{4} A_i)^2}{\prod_{i=1}^{4} A_i^2}. \tag{28}$$

The measure is therefore given by

$$d\mu(l_1, \ldots, l_6) = \text{constant} \times \frac{\sum_{i=1}^{4} A_i}{\prod_{i=1}^{4} A_i} \prod_{j=1}^{6} l_j \, dl_j \, \delta(\triangle). \tag{29}$$

† We remark that precisely in $D = 1$ one cannot use the canonical value $C = -2$. From the point of view of the continuum path integral this is not important since one can just choose a different value of C.
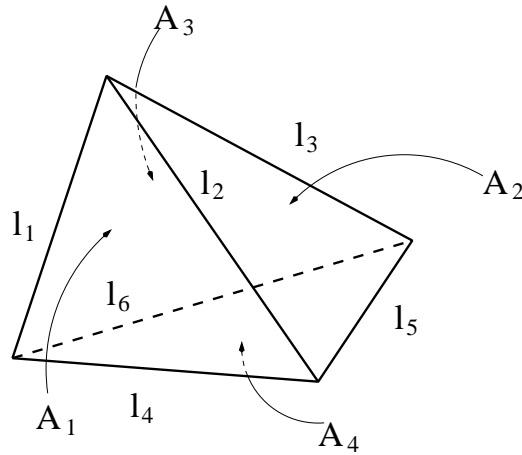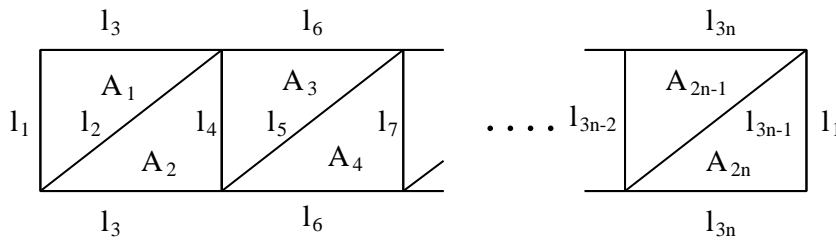
Figure 3. Conventions for the tetrahedron.

Figure 4. Conventions for the $1 \times n$-torus.

Unfortunately the simple beauty of this formula does not extend to more complicated geometries.

2.3.3. $1 \times n$-torus.   It is also possible to compute the measure for a thin torus explicitly. The $1 \times n$-torus is made of $3n$ links and $2n$ areas, see figure 4. Because the links l_{3k-1} and l_{3k} are only coupled to links inside one block of two triangles, the $3n \times 3n$-matrix $\mathbf{M}$ has a block-diagonal form, see figure 5. The block number k has the following form:

$$\begin{bmatrix} \frac{1}{A_{2(k-1)}} + \frac{1}{A_{2k-1}} & -\frac{1}{A_{2k-1}} & -\frac{1}{A_{2k-1}} & 0 \\ -\frac{1}{A_{2k-1}} & \frac{1}{A_{2k-1}} + \frac{1}{A_{2k}} & -\frac{1}{A_{2k-1}} - \frac{1}{A_{2k}} & -\frac{1}{A_{2k}} \\ -\frac{1}{A_{2k-1}} & -\frac{1}{A_{2k-1}} - \frac{1}{A_{2k}} & \frac{1}{A_{2k-1}} + \frac{1}{A_{2k}} & -\frac{1}{A_{2k}} \\ 0 & -\frac{1}{A_{2k}} & -\frac{1}{A_{2k}} & \frac{1}{A_{2k}} + \frac{1}{A_{2k+1}} \end{bmatrix}. \tag{30}$$

By making simple row and column transformations on the matrix $\mathbf{M}$ one can extract a factor $4^n \prod_{k=1}^{n} (\frac{1}{A_{2k-1}} + \frac{1}{A_{2k}})$ from the determinant of $\mathbf{M}$. We expand the remaining determinant using its linearity in the rows and in the columns. The result is

$$\det \mathbf{M} = (-4)^n \prod_{k=1}^{n} \left(\frac{1}{A_{2k-1}} + \frac{1}{A_{2k}} \right) \left(\prod_{k=1}^{n} \frac{1}{A_{2k-1}} + (-1)^{n-1} \prod_{k=1}^{n} \frac{1}{A_{2k}} \right)^2. \tag{31}$$
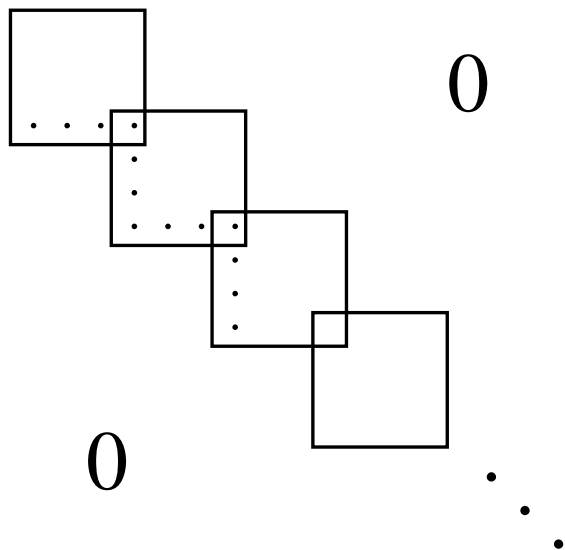
Figure 5. Block-diagonal form of the matrix **M**.

Thus the DeWitt-measure for the $1 \times n$-torus is

$$d\mu(l_1, \ldots, l_{3n}) = \text{constant} \times \frac{\prod_{j=1}^{3n} l_j \, dl_j \, \delta(\triangle)}{\prod_{k=1}^{2n} A_k^{3/2}} \prod_{k=1}^{n} (A_{2k-1} + A_{2k})^{1/2}$$

$$\times \left| \prod_{k=1}^{n} A_{2k} + (-1)^{n-1} \prod_{k=1}^{n} A_{2k-1} \right|. \tag{32}$$

Note that the measure of this special geometry might vanish if n is even. This happens for instance if all triangles have the same area.

2.3.4. Hedgehog geometry. As a last example we compute the DeWitt measure for a somewhat singular 'hedgehog' geometry which has played an important role in the analysis of the DT version of 2d quantum gravity coupled to matter fields [3]. The building blocks of the hedgehog geometry are 'hats' similar to the one shown in figure 6. This element consists of two triangles which are glued together along the links l_{2k-1} and l_{2k}. The areas of the triangles are A_{2k-1} and A_{2k}, respectively. Now we take N such elements and glue them together successively along the links l_{b_k} and $l_{a_{k+1}}$ to make a hedgehog-like geometry. After the gluing the vertices x and y will have the order $2N$, whereas the vertices v_k have the order two.

If we now write down the rows and columns of **M** in the order $l_{a_k}, l_{2k-1}, l_{2k}, l_{b_k}, \ldots$, we see that **M** takes indeed the same form as the matrix **M** for the thin torus in the last section. Therefore the measure will also be given by equation (32).
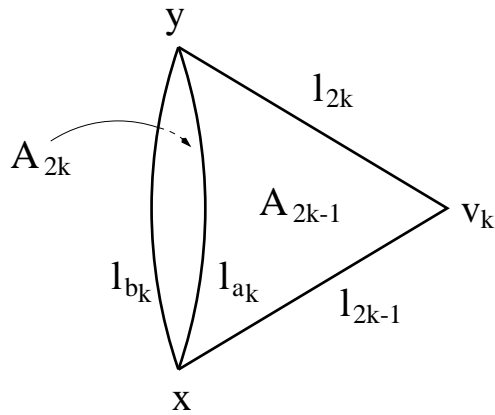
Figure 6. Elements of the hedgehog.

3. Diseases of quantum Regge calculus

Let us recall some of the results from continuum 2d quantum gravity based on the functional integral (1) in the case of manifolds with no handles†. The partition function is

$$Z(\Lambda) = \Lambda^{-5/2}, \tag{33}$$

and the Hartle–Hawking wavefunctional for a universe with boundary which has length ℓ and one marked point is given by

$$W(\ell, \Lambda) = \frac{1}{\ell^{5/2}}(1 + \ell\sqrt{\Lambda})e^{-\ell\sqrt{\Lambda}}. \tag{34}$$

Finally, let us consider (closed) universes of fixed volume V (i.e. fixed area since we consider 2d universes). The (suitable normalized) partition function for such universes, where in addition two marked points are separated by a geodesic distance R, is given by

$$Z(R, V) = V^{-1/4}F(R/V^{\frac{1}{4}}), \tag{35}$$

where $F(x)$ can be expressed in terms of certain hypergeometric functions [10, 11]. Here we only need the fact that $F(x)$ behaves as

$$F(x) \sim e^{-x^{4/3}} \qquad \text{for } x \to \infty. \tag{36}$$

From (35) and (36) it follows that *any* power of the average radius can be calculated as

$$\langle R^n \rangle_V = \int_0^\infty d\tilde{R}\, Z(\tilde{R}, V)\tilde{R}^n \sim V^{n/4}. \tag{37}$$

This relation shows that the fractal dimension of 2d quantum gravity is equal to four! [12, 10].

While we are unable to calculate (33) and (34) in QRC, we can show that (37) is not satisfied in QRC. In fact, we will prove that in general (37) is not defined in QRC:

† Since all critical indices which can be calculated in continuum quantum gravity, using either the KPZ or the DDK formalism, agrees with the critical indices which can be calculated using the DT formalism, we consider the two as equivalent. There are a number of results which can only be obtained using the DT formalism. We will still denote these as 'continuum results'.
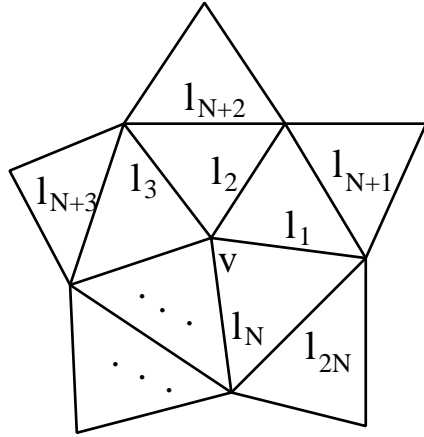
Figure 7. Parametrization of the link lengths.

Theorem. For any α or β in (21), (22) and (23) there exists an n such that for any link l in a given triangulation and any (fixed) value of the spacetime volume

$$\langle l^n \rangle_V = \infty. \tag{38}$$

As a consequence of this theorem the average radius $\langle R \rangle$, or some suitable power $\langle R^n \rangle$, does not exist. This is in sharp contrast to the situation encountered in equation (37). It shows that the ensemble average over geometries defined in QRC has no genuine intrinsic scale set by the volume of spacetime or equivalently by the cosmological constant. In this way the theory differs radically from the corresponding continuum quantum gravity theory.

Proof. Let the link l_1 be connected to a vertex v of coordination number N, see figure 7. We want to analyse the situation where the vertex v is pulled to infinity while the global area of the surface is held bounded. This can be done by keeping the link lengths $l_{N+1}, \ldots, l_{2N}$ of the order l_1^{-1}. Around v the link length l_1 can be integrated freely from a large length L to infinity. Then the integration of $l_2, \ldots, l_N$ is constrained by triangle inequalities. For the measure (22) the integration over $l_2, \ldots, l_N$ yields $(l_{N+2} \ldots l_{2N})^{\frac{N-1}{N}} l_1^{(1-N)\alpha}$. Here we symmetrized over the links $l_{N+1}, \ldots, l_{2N}$. The functional integral in the configuration space corresponding to this situation around the vertex v is then given by

$$\int_L^\infty \mathrm{d}l_1 \, l_1^{-N\alpha} \int_0^{L/l_1} \mathrm{d}l_{N+1} \ldots \mathrm{d}l_{2N} (l_{N+1} \ldots l_{2N})^{\frac{2N-1}{N} - \alpha}. \tag{39}$$

The additional factor $l_{N+1} \ldots l_{2N}$ in (39) originates from the triangle inequalities for the adjacent triangles. The integral over $l_{N+1}, \ldots, l_{2N}$ exists if $\frac{2N-1}{N} - \alpha > -1$, i.e.

$$\alpha < 3 - \frac{1}{N}. \tag{40}$$

Thus the l_1 integration becomes

$$\int_L^\infty \mathrm{d}l_1 \, l_1^{-N\alpha} \left(\frac{L}{l_1} \right)^{N\left(\frac{3N-1}{N} - \alpha \right)} \propto \int_L^\infty \mathrm{d}l_1 \, l_1^{1-3N}, \tag{41}$$

This shows that $\langle l_1^{3N-2} \rangle = \infty$ which completes the proof for the measure (22). In a similar way one can analyse the two other measures (23) and (21). To this end we have to
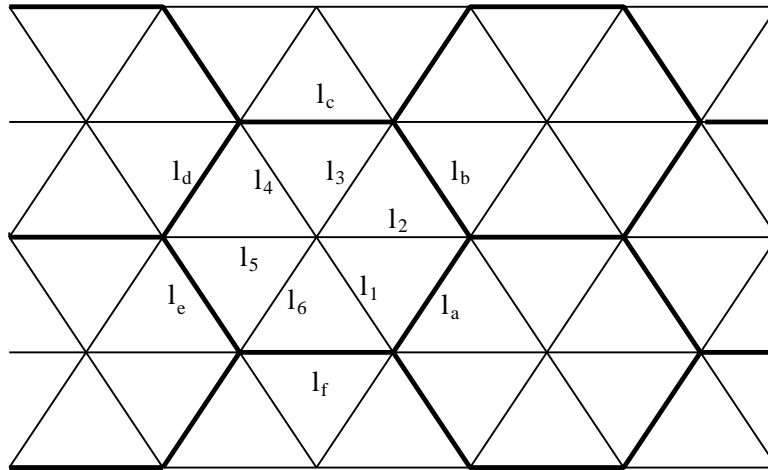
Figure 8. Only spikes in a hexagonal geometry.

parametrize the areas in terms of the link lengths. In this part of the configuration space the areas are given by the product of a small and long link length leading to extra factors $l_1^{-\beta} \ldots l_N^{-\beta}$ times $l_{N+1}^{-2\beta} \ldots l_{2N}^{-2\beta}$. By performing the same analysis as above for (23) we see that the functional integral is proportional to

$$\int_L^\infty \mathrm{d}l_1 \, l_1^{1-3N+N\beta}. \tag{42}$$

Thus $\langle l_1^n \rangle = \infty$ for $n \geqslant N(3-\beta) - 2$.

For (21) one can extract a factor of $\prod_{i=1}^L l_i^{1/2}$ from the nonlocal factor $\sqrt{P}$. Performing the analysis as above we again obtain the result (42). $\qquad\square$

In the following we will analyse (38) for a number of different triangulations in order to illustrate how sensitive the QRC measures are to the choice of triangulation, which is another worrisome aspect of this formalism. For some triangulations the measure itself is ill defined, for some triangulations $\langle l \rangle = \infty$ while for other measures the expectation value of some link l may be finite while the expectation value of l^2 is infinite etc. We denote $\langle l^n \rangle = \infty$ as the *appearance of spikes*.

3.1. Spikes in a hexagonal geometry

Let us first analyse the case of a hexagonal triangulation. We subdivide a regular triangulation as in figure 8 into N hexagonal cells. We analyse the functional integral in that part of the configuration space, where all centres of the cells form spikes. This means that the $6N$ links $l_1, \ldots, l_6$, etc. are very large. Furthermore, because the action is proportional to the area of the whole surface, we have to keep the area bounded to prevent exponential damping. Therefore the $3N$ link lengths $l_a, \ldots, l_f$, etc have to be very small, of the order of l_1^{-1}.

In cell number k one link, l_{6k+1}, can be integrated freely from a large length L to ∞. The integration of $l_{6k+2}, \ldots, l_{6k+6}$ is then constrained by triangle inequalities. For the measure (22) the integration over $l_{6k+2}, \ldots, l_{6k+6}$ thus yields a factor $(l_{a_k} \ldots l_{f_k})^{\frac{5}{6}} l_{6k+1}^{-5\alpha}$ in the kth cell. Here we have symmetrized over all the links $l_{a_k}, \ldots, l_{f_k}$ of the kth cell. Noting

that we can assign the three links $l_{a_k}, l_{b_k}, l_{c_k}$ to the kth cell, the functional integral in the spiky part of the configuration space is proportional to

$$\int_L^\infty \prod_{k=1}^N l_{6k+1}^{-6\alpha} \, \mathrm{d}l_{6k+1} \int_0^{L/l_{6k+1}} \prod_{k=1}^N (l_{a_k} l_{b_k} l_{c_k})^{\frac{5}{3}-\alpha} \, \mathrm{d}l_{a_k} \, \mathrm{d}l_{b_k} \, \mathrm{d}l_{c_k}. \tag{43}$$

The integration of the links $l_{a_k}, l_{b_k}, l_{c_k}$ only exists if

$$\alpha < \tfrac{8}{3}. \tag{44}$$

If we perform the integration, an integral of the following form will remain:

$$\int_L^\infty \prod_{k=1}^N l_{6k+1}^{-3\alpha-8} \, \mathrm{d}l_{6k+1}, \tag{45}$$

which only exists if

$$\alpha > -\tfrac{7}{3}. \tag{46}$$

Furthermore, from (45) we see that the expectation value of l_1^n will be infinite if n is larger than or equal to $7 + 3\alpha$.

If we analyse the measure (23) along the same lines, we have to parametrize the areas of the triangles in terms of the link lengths. For a spiky geometry the areas are given by products of a small and a long link in the triangle. We thus get extra factors of $l_{6k+1}^{-\beta} \ldots l_{6k+6}^{-\beta}$ and $l_{a_k}^{-2\beta} \ldots l_{f_k}^{-2\beta}$. Then we can perform the analysis as above and obtain the bound

$$\beta < \tfrac{11}{6} \tag{47}$$

for β. In this case we get no lower bound. The expectation value of l_1^n will be infinite if $n \geqslant 4$.

For this geometry one can extract a factor $\prod_{i=1}^L l_i^{1/2}$ from the nonlocal factor $\sqrt{P}$ of the measure (21). Consequently, we get for this measure extra factors of $l_{6k+1}^{\frac{1}{2}-\beta} \ldots l_{6k+6}^{\frac{1}{2}-\beta}$ and $l_{a_k}^{\frac{1}{2}-2\beta} \ldots l_{f_k}^{\frac{1}{2}-2\beta}$ from the areas. Using this we obtain the bound

$$\beta < \tfrac{25}{12} \tag{48}$$

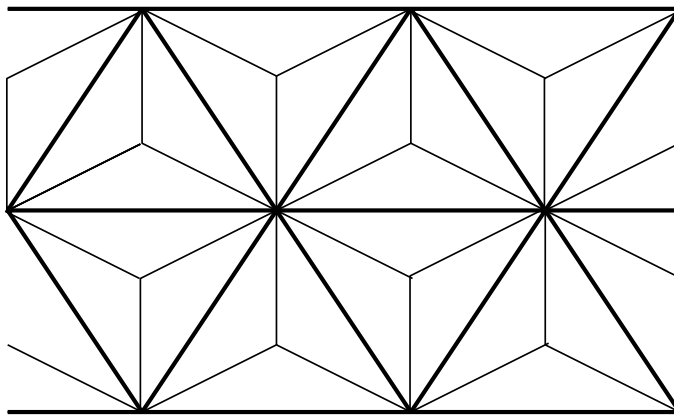and the expectation value of l_1^n will be infinite for $n \geqslant \tfrac{5}{2}$.

Figure 9. A 12–3 geometry. All vertices of order 3 form spikes.

Table 1. Bounds for the exponents α and β for the measures (22), (23) and (21) and values of n for which $\langle l^n \rangle$ is finite for the 12–3 geometry.

Measure	Bounds	Values of n s.t. $\langle l^n \rangle < \infty$
(22)	$-\frac{5}{3} < \alpha < \frac{7}{3}$	$n < \frac{5}{2} + 3\alpha$
(23)	$\beta < \frac{5}{3}$	$n < 1$
(21)	$\beta < \frac{23}{12}$	$n < \frac{1}{4}$

Table 2. The upper part of the table shows bounds for the exponents α and β for the measures (22), (23) and (21) and values of n for which $\langle l^n \rangle$ is finite in the case illustrated in the second part of figure 10. The lower part of table shows the similar bounds for hedgehog geometries constructed from the building blocks illustrated in figure 6.

Measure	Bounds	Values of n s.t. $\langle l^n \rangle < \infty$
(22)	$-2 < \alpha < \frac{5}{2}$	$n < 2\alpha + 4$
(23)	$\beta < \frac{7}{4}$	$n < 2$
(21)	$\beta < 2$	$n < 1$
(22)	$-1 < \alpha < 2$	$n < 1 + \alpha$
(23)	Partition function undefined	$n < 0$
(21)	Partition function undefined	$n < -\frac{1}{2}$

3.2. Spikes in a 12–3 geometry

In a similar way we can analyse the partition function for a regular toroidal triangulation where N vertices have order 3 and $N/2$ vertices have order 12, see figure 9. We can form spikes by pulling the vertices of order 3 to infinity keeping the area bounded from above. The bounds for the measures (22), (23) and (21) are given in table 1. Note that these bounds are sharper than for the hexagonal geometry.

3.3. Spikes in hedgehog-like geometries

One can get sharper bounds for more degenerate geometries. As explained above, these kind of geometries occur in several phases of quantum gravity in the context of DT. However, the considerations are essentially local and for very large triangulations it would be an unnatural constraint on the possible choices of triangulations if hedgehog-like connectivity should not be allowed locally. As a first example we analyse the geometry depicted in figure 10. Here there are two vertices of order N while all other vertices have order 4. This is a slightly more regular configuration than the hedgehog geometry associated with figure 6.

If spikes are formed by every second vertex of order 4, see the second part of figure 10, the bounds take the form given in upper part of table 2.

For a hedgehog configuration constructed from the spikes in figure 6 we get the following bounds recorded in the lower part of table 2, when the spike-vertices are allowed to go to infinity.

3.4. One vanishing link

Besides spikes one can also analyse the behavior of the functional integral if one or more links go to zero as in figure 11. In the notation of figure 11 we demand that l_a is small
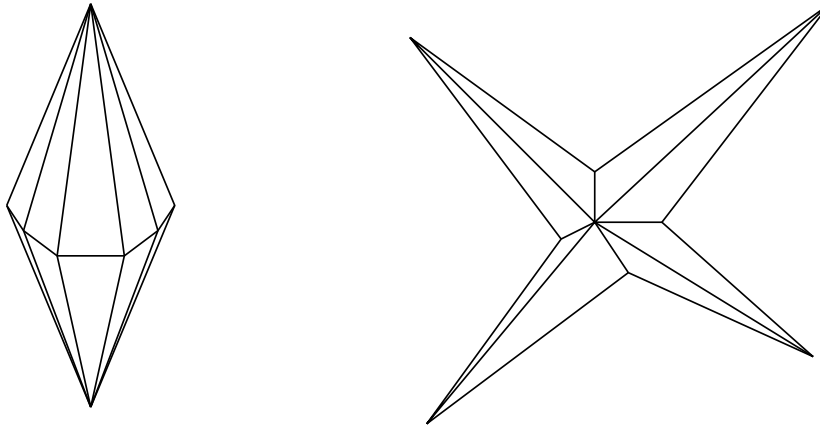
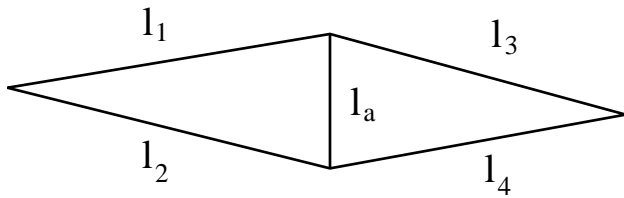**Figure 10.** A degenerate geometry. Spikes can be formed in several ways.

Figure 11. If one link length (l_a) vanishes, the others stay finite. The whole area is again bounded from above.

Table 3. Bounds on the exponents α and β for the case of one vanishing link length.

Measure	Bound on exponent
(22)	$\alpha < 3$
(23)	$\beta < 2$
(21)	$\beta < \frac{9}{4}$

whereas the other link lengths are bounded away from zero and infinity. Then one can integrate the links l_1 and l_3 freely and the integration of l_2 and l_4 is constrained by triangle inequalities. Their integration therefore yields a factor l_a^2 and some powers of l_1 and l_3 which depend on the measures. The resulting bounds are given in table 3.

4. Conclusion

We have shown that the predictions of QRC do not agree with known results of continuum 2d quantum gravity, as defined by the functional (1). No natural scale seems to emerge from measures (22), (21) and even expectation values of suitable powers of the length of a *single* link will diverge. The probability of having arbitrary large link length will never be exponentially suppressed even if the volume of spacetime is kept bounded. This is in contrast to the situation for continuum 2d quantum gravity where the probability of having

two points separated by a distance R is exponentially suppressed as

$$\exp(-R\sqrt{\Lambda})$$

for given cosmological constant, and suppressed as

$$\exp\left\{-\left(\frac{R}{V^{\frac{1}{4}}}\right)^{4/3}\right\}$$

if the volume of spacetime is kept fixed to be V [10].

In addition the finiteness of $\langle l^n \rangle$ for given values of α and β (and small n where the expectation value has a chance to exist) depends crucially on the chosen fixed triangulation in QRC. In fact, for given values of α and β the bare existence of the partition function could depend on the chosen triangulation. Clearly this is not a desirable situation.

We expect the situation to be similar in higher dimensional QRC. The measures (22) and (23) have an obvious generalization to higher dimensional simplicial manifolds. The DeWitt-like measure for a D-dimensional simplex can be obtained from the norm

$$\|\delta g_{\mu\nu}\|_D^2 = \delta x \mathbf{M}_D \delta x, \tag{49}$$

where δx is the $D(D+1)/2$ dimensional vector of $x_i = l_i^2$ deviations. As in the 2d case the line element for the D-dimensional triangulation is the sum of line elements (49) for the individual simplexes constituting the triangulation. Thus the structure of the $\mathbf{M}_D^{\text{total}}$ is an obvious superposition of the basic matrices (49) in a way similar to the 2d case (see (15)).

For instance, in $D = 3$ the matrix $\mathbf{M}_D$ is six dimensional (6d) with entries depending on x_i's. The measure

$$d\mu = \sqrt{\det \mathbf{M}_D^{\text{total}}} \prod_{i=1}^{L} dx_i \, \delta(\triangle) \tag{50}$$

for the closed manifold, constructed from two tetrahedra glued together, is equal to

$$d\mu(l_1, \ldots, l_6) = \frac{\text{constant}}{V} \prod_{i=1}^{6} l_i \, dl_i \, \delta(\triangle), \tag{51}$$

where V is the tree volume of the manifold. Thus the partition function of the two-simplex gravity is given by

$$\int e^{-\Lambda V + \sum_{i=1}^{6} l_i(2\pi - 2\alpha_i)} \delta(\triangle) \frac{1}{V} \prod_{i=1}^{6} l_i \, dl_i \tag{52}$$

where α_i are the dihedral angles. (51) is the three-dimensional (3d) expression corresponding to (27). Already the partition function corresponding to (27) is divergent (see table 2) and producing spikes and (52) is even less well defined due to unboundedness of the action in three dimensions. It is well known that this disease has to be cured by a cut-off involving for instance higher derivative terms (for a definition in the context of integral geometry, see [13]). However, the problem with spikes will reappear as the cut-off is taken to zero and even if we completely drop the curvature term from the action the partition function will not be well defined, precisely as in the 2d case. Although we have not investigated higher dimensional cases in detail it seems hard to believe that QRC is a viable candidate for the quantum gravity before one is able to find the correct measure able of reproducing the results of the simplest known such theory, namely 2d quantum gravity.

Acknowledgments

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