

Anosov-Kolmogorov CK systems and MIXMAX PRNG for MC simulations

George Savvidy

Demokritos National Research Center
Athens, Greece

Integration of the MIXMAX engine into the CERN
Scientific Software ROOT and Geant4

3 May 2017
Athens

MIXMAX random number generators

Anosov C-systems

Spectrum and Kolmogorov Entropy of the C-systems

A(N,s) and A(N,s,m) Family of C-operators

- 1.G.Savvidy and Natalia Savvidi,
On the Monte Carlo simulation of physical systems
J.Comput.Phys. **97** (1991) 566; Preprint EFI, 1986
- 2.K.Savvidy, The MIXMAX random number generator
Comput.Phys.Commun. 196 (2015) 161
- 3.G. Savvidy, Anosov C-systems and Random Number Generators
Theor.Math.Phys. 2016; arXiv:1507.06348
- 4.K.Savvidy and G.Savvidy
Spectrum and Entropy of C-systems. MIXMAX Generator,
Chaos, Solitons and Fractals, 91 (2016) 33-38
- 5.G. Savvidy and K. Savvidy,
Hyperbolic Anosov C-systems.
Exponential Decay of Correlation Functions,
arXiv:1702.03574 [math-ph]

- ▶ HEPFORGE.ORG, <http://mixmax.hepforge.org>;
<http://www.inp.demokritos.gr/savvidy/mixmax.php>

- ▶ HEPFORGE.ORG, <http://mixmax.hepforge.org>;
<http://www.inp.demokritos.gr/savvidy/mixmax.php>
- ▶ ROOT, Release 6.04/06 on 2015-10-13,
https://root.cern.ch/doc/master/mixmax_8h_source.html

- ▶ HEPFORGE.ORG, <http://mixmax.hepforge.org>;
<http://www.inp.demokritos.gr/savvidy/mixmax.php>
- ▶ ROOT, Release 6.04/06 on 2015-10-13,
https://root.cern.ch/doc/master/mixmax_8h_source.html
- ▶ CLHEP, Release 2.3.1.1, on November 10th, 2015
<http://proj-clhep.web.cern.ch/proj-clhep/>

mixmax realese 110

speed = $4 \cdot 10^{-9}$ second

mixmax.c

mixmax.h

driver main.c

driver verification.c

mixmax skip N240.c

mixmax skip N17.c

-- -- > README.pdf

mixmax gsl driver.c

driver testU01.c

Size N	Magic s	Entropy	Period $\approx \log_{10}(q)$
256	-1	164.5	4682
256	487013230256099064	193.6	4682

Table : Properties of operators $A(N,s)$ for different special s .

Size N	Magic m	Magic s	Entropy	Period $\approx \log_{10}(q)$
8	$m = 2^{53} + 1$	$s=0$	220.4	129
17	$m = 2^{36} + 1$	$s=0$	374.3	294
40	$m = 2^{42} + 1$	$s=0$	1106.3	716
60	$m = 2^{52} + 1$	$s=0$	2090.5	1083
96	$m = 2^{55} + 1$	$s=0$	3583.6	1745
120	$m = 2^{51} + 1$	$s=1$	4171.4	2185
240	$m = 2^{51} + 1$	$s=487013230256099140$	8418.8	4389

Table : Table of three-parameter MIXMAX generators $A(N,s,m)$. These generators have an advantage of having a very high quality sequence for moderate and small N . In particular, the smallest generator we tested, $N = 8$, passes all tests in the BigCrush suite.

Size N	Magic s	Entropy (lower bound)	Period $\approx \log_{10}(q)$
7307	0	4502.1	134158
20693	0	12749.5	379963
25087	0	15456.9	460649
28883	1	17795.7	530355
40045	-3	24673.0	735321
44851	-3	27634.1	823572

Table : Table of properties of the operator $A(N, s)$ for large matrix size N . The third column is the value of the Kolmogorov entropy. All these generators passes tests in the BigCrush suite. For the largest of them the period approaches a *million digits*.

Dmitri Anosov, in his fundamental work on *hyperbolic dynamical C-systems* pointed out that the basic property of the geodesic flow on closed Riemannian manifolds V^n of negative curvature is a *uniform instability of all its trajectories*.

In physical terms that means that *in the neighbourhood of every fixed trajectory the trajectories behave similarly to the trajectories in the neighbourhood of a saddle point*.

The hyperbolic instability of the dynamical system $\{T^t\}$ which is defined by the equations ($w \in W^m$)

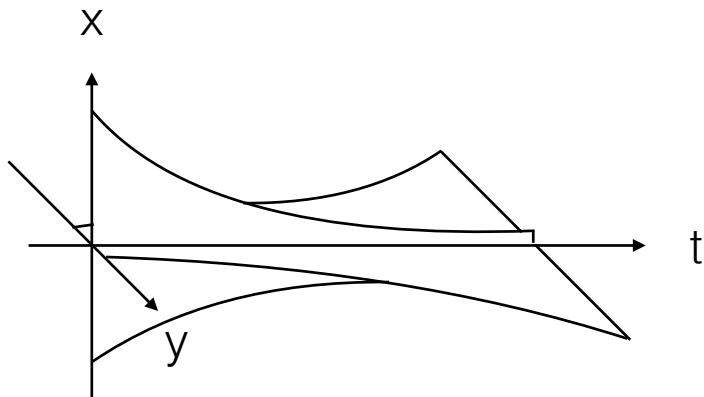
$$\dot{w} = f(w) \quad (1)$$

takes place for all solutions $\delta w \equiv \omega$ of the deviation equation

$$\dot{\omega} = \left. \frac{\partial f}{\partial w} \right|_{w(t)=T^t w} \omega \quad (2)$$

in the neighbourhood of each phase trajectory $w(t) = T^t w$.

The Anosov C-systems are genuine hyperbolic systems



the behaviour of all nearby trajectories is exponentially unstable

The C-condition requires that the tangent space R_w^m at each point w of the m -dimensional phase space W^m of the dynamical system $\{T^n\}$ should be decomposable into a direct sum of the two linear spaces X_w^k and Y_w^l with the following properties:

C1.

$$R_w^m = X_w^k \oplus Y_w^l$$

and they are such that $\{\tilde{T}^n\}$

C2.

$$a) |\tilde{T}^n \xi| \leq a |\xi| e^{-cn} \text{ for } n \geq 0; |\tilde{T}^n \xi| \geq b |\xi| e^{-cn} \text{ for } n \leq 0, \xi \in X_w^k$$

$$b) |\tilde{T}^n \eta| \geq b |\eta| e^{cn} \text{ for } n \geq 0; |\tilde{T}^n \eta| \leq a |\eta| e^{cn} \text{ for } n \leq 0, \eta \in Y_w^l,$$

where the constants a, b and c are positive and are the same for all $w \in W^m$ and all $\xi \in X_w^k, \eta \in Y_w^l$. The length $|\dots|$ of the tangent vectors ξ and η is defined by the Riemannian metric ds on W^m .

The contracting and expanding foliations Σ_w^k and Σ_w^l

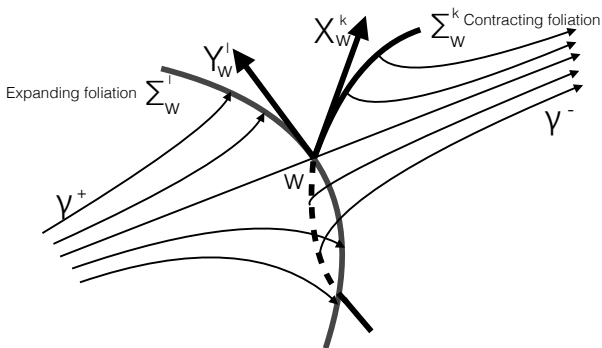


Figure : At each point w of the C-system the tangent space R_w^m is decomposable into a direct sum of two linear spaces Y_w^l and X_w^k . The expanding and contracting geodesic flows are γ^+ and γ^- . The expanding and contracting invariant foliations Σ_w^l and Σ_w^k are transversal to the geodesic flows and their corresponding tangent spaces are Y_w^l and X_w^k .

Important Example of C-system: *Torus Automorphisms*

Consider linear automorphisms of the unit hypercube in Euclidean space R^N with coordinates (u_1, \dots, u_N) where $u \in [0, 1)$

$$u_i^{(k+1)} = \sum_{j=1}^N A_{ij} u_j^{(k)}, \quad \text{mod } 1, \quad k = 0, 1, 2, \dots \quad (6)$$

- ▶ The dynamical system defined by the integer matrix A has determinant equal to one $\text{Det}A = 1$.

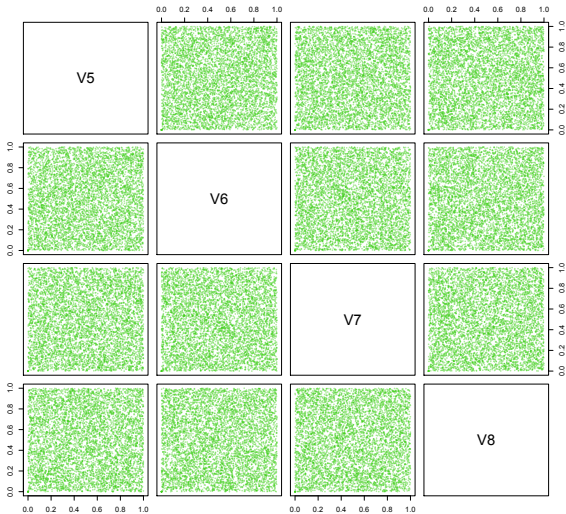
Important Example of C-system: *Torus Automorphisms*

Consider linear automorphisms of the unit hypercube in Euclidean space R^N with coordinates (u_1, \dots, u_N) where $u \in [0, 1)$

$$u_i^{(k+1)} = \sum_{j=1}^N A_{ij} u_j^{(k)}, \quad \text{mod } 1, \quad k = 0, 1, 2, \dots \quad (6)$$

- ▶ The dynamical system defined by the integer matrix A has determinant equal to one $DetA = 1$.
- ▶ The Anosov hyperbolicity C-condition: the matrix A has no eigenvalues on the unit circle. Thus the spectrum $\Lambda = \lambda_1, \dots, \lambda_N$ fulfils the two conditions:

$$1) DetA = \lambda_1 \lambda_2 \dots \lambda_N = 1, \quad 2) |\lambda_i| \neq 1. \quad (7)$$



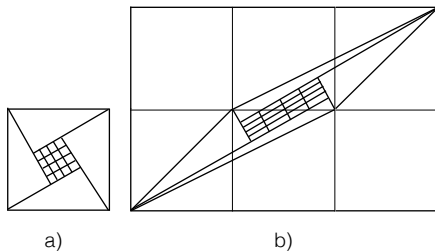
- ▶ The eigenvalues of the matrix A are divided into the two sets $\{\lambda_\alpha\}$ and $\{\lambda_\beta\}$ with modulus smaller and larger than one:

$$0 < |\lambda_\alpha| < 1 < |\lambda_\beta|. \quad (8)$$

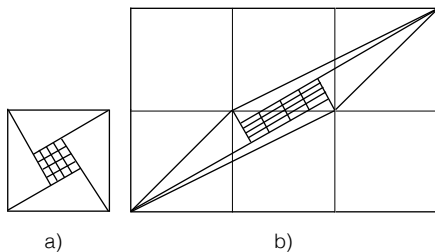
- ▶ The eigenvalues of the matrix A are divided into the two sets $\{\lambda_\alpha\}$ and $\{\lambda_\beta\}$ with modulus smaller and larger than one:

$$0 < |\lambda_\alpha| < 1 < |\lambda_\beta|. \quad (8)$$

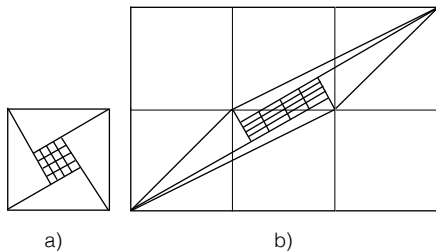
- ▶ There exist two families of planes $X = \{X_\alpha\}$ and $Y = \{Y_\beta\}$ which are parallel to the corresponding eigenvectors $\{e_\alpha\}$ and $\{e_\beta\}$.



- ▶ The eigenvectors of the matrix A $\{e_\alpha\}$ and $\{e_\beta\}$ define two families of parallel planes $\{X_\alpha\}$ and $\{Y_\beta\}$



- ▶ The eigenvectors of the matrix A $\{e_\alpha\}$ and $\{e_\beta\}$ define two families of parallel planes $\{X_\alpha\}$ and $\{Y_\beta\}$
- ▶ The automorphism A is contracting the points on the planes $\{X_\alpha\}$ and expanding points on the planes $\{Y_\beta\}$.



- ▶ The eigenvectors of the matrix A $\{e_\alpha\}$ and $\{e_\beta\}$ define two families of parallel planes $\{X_\alpha\}$ and $\{Y_\beta\}$
- ▶ The automorphism A is contracting the points on the planes $\{X_\alpha\}$ and expanding points on the planes $\{Y_\beta\}$.
- ▶ The a) depicts the parallel planes of the sets $\{X_\alpha\}$ and $\{Y_\beta\}$ and b) depicts their positions after the action of the automorphism A.

- ▶ The Kolmogorov entropy of a Anosov C-system is:

$$h(A) = \sum_{|\lambda_\beta|>1} \ln |\lambda_\beta|. \quad (9)$$

The entropy $h(A)$ depends on the spectrum of the operator A .

- ▶ The Kolmogorov entropy of a Anosov C-system is:

$$h(A) = \sum_{|\lambda_\beta|>1} \ln |\lambda_\beta|. \quad (9)$$

The entropy $h(A)$ depends on the spectrum of the operator A .

- ▶ *This allows to characterise and compare the chaotic properties of dynamical C-systems quantitatively → computing and comparing their entropies.*

- ▶ The decorrelation time τ_0 of the dynamical system A can be expressed in terms of its entropy

$$\tau_0 = 1/h(A) \quad (10)$$

- ▶ The decorrelation time τ_0 of the dynamical system A can be expressed in terms of its entropy

$$\tau_0 = 1/h(A) \quad (10)$$

- ▶ The relaxation time τ , at which the initial states occupying the volume δv_0 will spread over the whole phase space, is

$$\tau = \tau_0 \ln(1/\delta v_0) \quad (11)$$

- ▶ The decorrelation time τ_0 of the dynamical system A can be expressed in terms of its entropy

$$\tau_0 = 1/h(A) \quad (10)$$

- ▶ The relaxation time τ , at which the initial states occupying the volume δv_0 will spread over the whole phase space, is

$$\tau = \tau_0 \ln(1/\delta v_0) \quad (11)$$

- ▶ These important characteristic time scales should fulfil the following fundamental relation

$$\tau_0 \leq T \leq \tau \quad (12)$$

where $T = 1$ is a time of one iteration.

Thus there are three characteristic time scales associated with the C-system:

$$\left(\begin{array}{c} \textit{Decorrelation} \\ \textit{time} \\ \tau_0 = \frac{\pi}{4pN^2} \end{array} \right) < \left(\begin{array}{c} \textit{Interaction} \\ \textit{time} \\ t_{int} = n = 1 \end{array} \right) < \left(\begin{array}{c} \textit{Stationary} \\ \textit{distribution time} \\ \tau = \frac{1}{h(T)} \ln \frac{1}{\delta v_0} \end{array} \right).$$

The generator $N = 256$ has the entropy $h(T) = 194$, therefore the characteristic time scales for this generator are

$$\left(\begin{array}{c} \textit{Decorrelation} \\ \textit{time} \\ \tau_0 = 0.000012 \end{array} \right) < \left(\begin{array}{c} \textit{Interaction} \\ \textit{time} \\ t_{int} = 1 \end{array} \right) < \left(\begin{array}{c} \textit{Stationary} \\ \textit{distribution time} \\ \tau = 95 \end{array} \right).$$

The MIXMAX generator $N = 240$ has the entropy $h(T) = 8679$, therefore the characteristic time scales for this generator are

$$\begin{pmatrix} \text{Decorrelation} \\ \text{time} \\ \tau_0 = 0.000004 \end{pmatrix} < \begin{pmatrix} \text{Interaction} \\ \text{time} \\ t_{int} = 1 \end{pmatrix} < \begin{pmatrix} \text{Stationary} \\ \text{distribution time} \\ \tau = 1.17 \end{pmatrix}.$$

Both generators have very short decorrelation time. The second generator $N = 240$ has much bigger entropy and therefore its relaxation time τ is much smaller, of order 1.17, and is close to the interaction time.

A strong instability of trajectories of a dynamical C-system leads to the appearance of statistical properties in its behaviour. As a result the time average of the function $f(w)$ on W^m

$$S_N(w) = \frac{1}{N} \sum_{n=0}^{N-1} f(A^n w) \quad (13)$$

behaves as a superposition of quantities which are statistically weakly dependent. Therefore for the C-systems on a torus it was demonstrated by Leonov that the *deviation of the time averages (13) from the phase space averages*

$$\langle f \rangle = \int_{W^m} f(w) dw \quad (14)$$

multiplied by \sqrt{N} have at large $N \rightarrow \infty$ the Gaussian distribution:

$$\lim_{N \rightarrow \infty} \mu \left\{ w : \sqrt{N} \left(S_N - \langle f \rangle \right) < z \right\} = \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^z e^{-\frac{y^2}{2\sigma_f^2}} dy.$$

The deviation of the time average S_N from its limit value $\langle f \rangle$ is asymptotically Gaussian! The quantity $\sqrt{N} \left(S_N(w) - \langle f \rangle \right)$ converges in distribution to the normal random variable with standard deviation σ_f

$$\sigma_f^2 = \sum_{n=-\infty}^{+\infty} [\langle f(w)f(A^n w) \rangle - \langle f(w) \rangle^2]. \quad (15)$$

We were able to express it in terms of entropy

$$\sigma_f^2 = \sum_{n=-\infty}^{+\infty} \frac{M^2}{128\pi^4} e^{-4nh(T)} = \frac{M^2}{128\pi^4} \frac{e^{4h(A)} + 1}{e^{4h(A)} - 1} \quad (16)$$

- ▶ The Anosov C-systems have very strong chaotic properties: the exponential instability of all trajectories
in fact the instability is as strong as it can be in principle.

- ▶ The Anosov C-systems have very strong chaotic properties: the exponential instability of all trajectories in fact the instability is as strong as it can be in principle.
- ▶ Our aim is to study these characteristics of the C-systems and develop our earlier suggestion to use the Anosov C-systems as random number generator for Monte-Carlo simulations

Family of operators $A(N,s)$ parametrised by the integers N and s

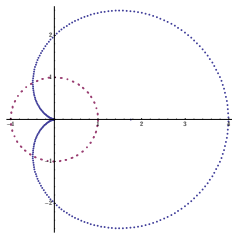
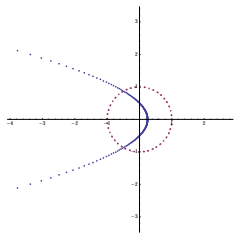
$$A(N,s) = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & 3+s & 2 & 1 & \dots & 1 & 1 \\ 1 & 4 & 3 & 2 & \dots & 1 & 1 \\ & & & \dots & & & \\ 1 & N & N-1 & N-2 & \dots & 3 & 2 \end{pmatrix} \quad (17)$$

The matrix is of the size $N \times N$

Its entries are all integers $A_{ij} \in \mathbb{Z}$

Det $A = 1$

The spectrum and the value of the Kolmogorov entropy?



Eigenvalue Distribution of $A(N,s)$ and of $A^{-1}(N,s)$
 all of them are lying outside of the unit circle

Size N	Magic s	Entropy	Period $\approx \log_{10}(q)$
256	-1	164.5	4682
256	487013230256099064	193.6	4682

Table : Properties of operators $A(N,s)$ for different special s .

Size N	Magic s	Entropy (lower bound)	Period $\approx \log_{10}(q)$
7307	0	4502.1	134158
20693	0	12749.5	379963
25087	0	15456.9	460649
28883	1	17795.7	530355
40045	-3	24673.0	735321
44851	-3	27634.1	823572

Table : Table of properties of the operator $A(N, s)$ for large matrix size N . The third column is the value of the Kolmogorov entropy. All these generators passes tests in the BigCrush suite. For the largest of them the period approaches a *million digits*.

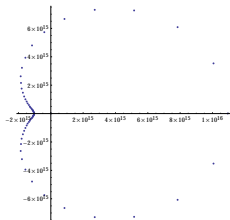
$A(N,s,m)$

A three-parameter family of C-operators $A(N, s, m)$, where m is some integer:

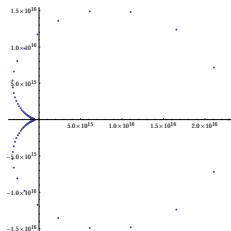
$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & m+2+s & 2 & 1 & \dots & 1 & 1 \\ 1 & 2m+2 & m+2 & 2 & \dots & 1 & 1 \\ 1 & 3m+2 & 2m+2 & m+2 & \dots & 1 & 1 \\ & & & \dots & & & \\ 1 & (N-2)m+2 & (N-3)m+2 & (N-4)m+2 & \dots & m+2 & 2 \end{pmatrix}$$

Size N	Magic m	Magic s	Entropy	Period $\approx \log_{10}(q)$
8	$m = 2^{53} + 1$	$s=0$	220.4	129
17	$m = 2^{36} + 1$	$s=0$	374.3	294
40	$m = 2^{42} + 1$	$s=0$	1106.3	716
60	$m = 2^{52} + 1$	$s=0$	2090.5	1083
96	$m = 2^{55} + 1$	$s=0$	3583.6	1745
120	$m = 2^{51} + 1$	$s=1$	4171.4	2185
240	$m = 2^{51} + 1$	$s=487013230256099140$	8418.8	4389

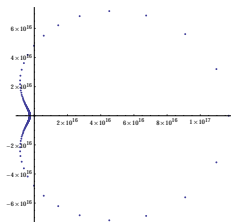
Table : Table of three-parameter MIXMAX generators $A(N,s,m)$. These generators have an advantage of having a very high quality sequence for moderate and small N . In particular, the smallest generator we tested, $N = 8$, passes all tests in the BigCrush suite.



The distribution of the eigenvalues of the $A(N, s, m)$ for $N = 60, s = 0, m = 2^{52} + 1$. The spectrum represents a leaf of a large radius proportional to $\lambda_{max} \approx m$ and a very small eigenvalue at the origin $\lambda_{min} \approx m^{-N+1}$.



The distribution of the eigenvalues of the $A(N, s, m)$ for $N = 120, s = 1, m = 2^{51} + 1$.



The distribution of the eigenvalues of the $A(N, s, m)$ for $N = 240, s = 487013230256099140, m = 2^{51} + 1$.

Conclusion

Use MIXMAX for your Monte-Carlo simulations !

it will provide a fast convergence!

This work was supported in part by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No 644121.

Thank you!