Integrand-level reduction with HELAC2LOOP

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H.F.R.I. Hellenic Foundation for Research & Innovation

HOCTools-II

Theory Challenges in the Precision Era of the Large Hadron Collider
Aug 28, 2023 - Oct 13, 2023

September 11, 2023

Outline

1. Introduction
2. HELAC2LOOP
3. Master Integrals
4. Loop-by-loop approach

Leading Order

How to avoid Feynman diagrams

$\rightarrow$ a highly subjective point of view

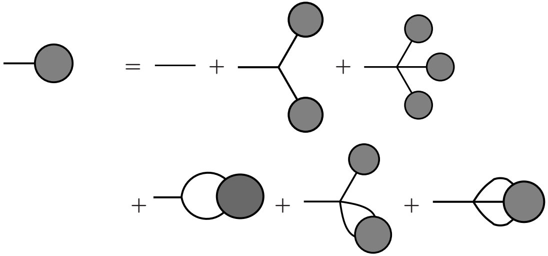
LO - Dyson-Schwinger Recursive Equations

MadGraph → T. Stelzer and W. F. Long, Comput. Phys. Commun. **81**, 357 (1994)

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

From Feynman Diagrams to recursive equations: taming the *n*!

- 1999 `HELAC`: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles



Unfortunately not so much on the second line !

LO - Dyson-Schwinger Recursive Equations

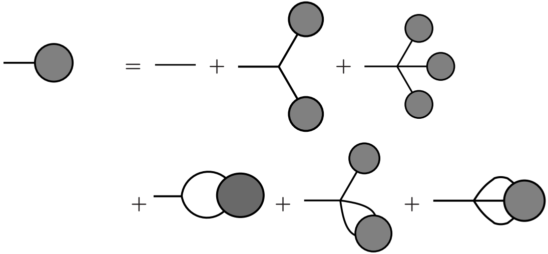
From Feynman Diagrams to recursive equations: taming the *n*!

- 1999 `HELAC`: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

→A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

→F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

→ F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

From Feynman graphs ...

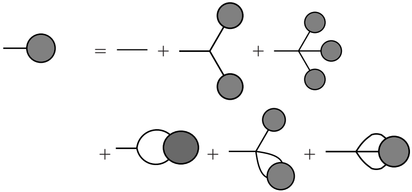
$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

Taming the beast ...

From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

to Dyson-Schwinger recursion! `Helac-Phegas`



$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

NLO

Don't make integrals, make integrands !

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \quad \leftarrow LO \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \quad \leftarrow Virtual \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \quad \leftarrow Real
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m^{D=4} \left(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV})) \right) J_m(\Phi) \\
&+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence μ_R

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}\sigma_{NLO} &= \int_m d\Phi_m J_m(\Phi) \\ &+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)\end{aligned}$$

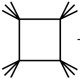
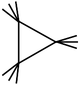
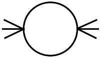
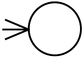
QCD factorization $-\mu_F$ Collinear counter-terms when PDF are involved basis of scalar integrals: known already before NLO-R; remember this is not the case for higher orders

→G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

→ Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751

→G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

→Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217.

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

THE OLD "MASTER" FORMULA

$$
\begin{aligned}
\mathcal{A} \to \int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "MASTER" FORMULA - I

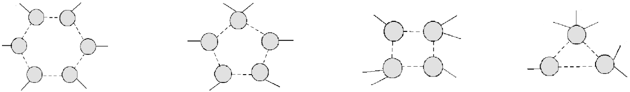
General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) \;=\; & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
+ & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
+ & \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

The one-loop calculation in a nutshell

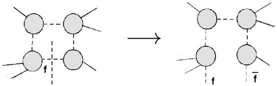
The computation of $pp(p\bar{p}) \to e^{+}\nu_e\mu^{-}\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots.$ with the values of the loop momentum q provided by CutTools

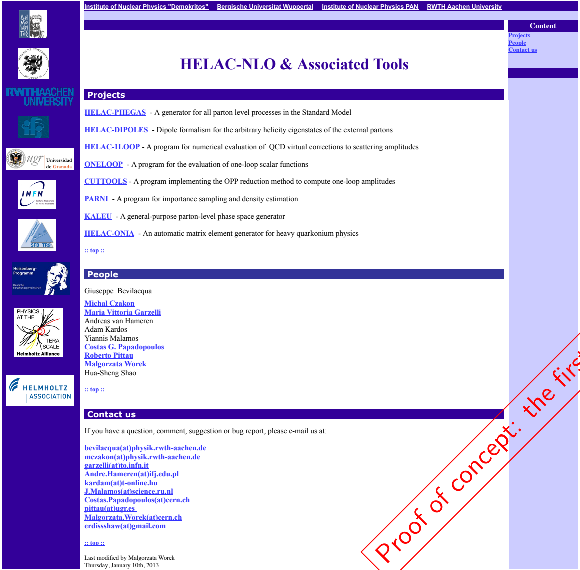
- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

→ BlackHat, MadGraph, RECOLA, OpenLoops

The one-loop calculation in a nutshell



Institute of Nuclear Physics "Demokritos" Bergische Universitat Wuppertal Institute of Nuclear Physics PAN RWTH Aachen University

HELAC-NLO & Associated Tools

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Projects

HELAC-PHEGAS - A generator for all parton level processes in the Standard Model

HELAC-DIPOLES - Dipole formalism for the arbitrary helicity eigenstates of the external partons

HELAC-1LOOP - A program for numerical evaluation of QCD virtual corrections to scattering amplitudes

ONELOOP - A program for the evaluation of one-loop scalar functions

CUTTOOLS - A program implementing the OPP reduction method to compute one-loop amplitudes

PARNI - A program for importance sampling and density estimation

KALEU - A general-purpose parton-level phase space generator

HELAC-ONIA - An automatic matrix element generator for heavy quarkonium physics

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People

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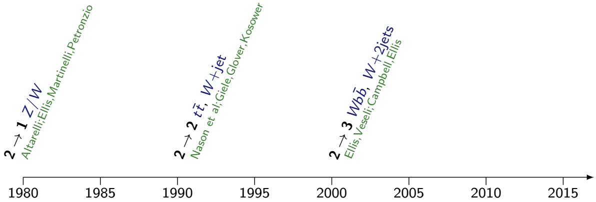
Last modified by Malgorzata Worek
Thursday, January 10th, 2013

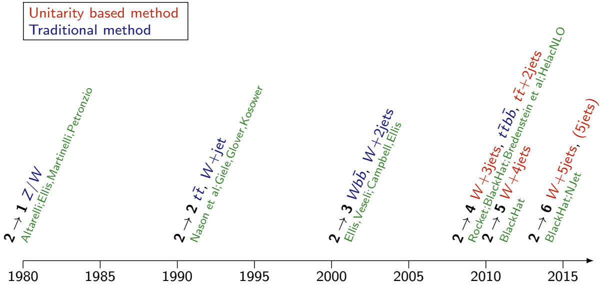
Proof of concept: the first NLO public code

G. P. Salam, PoS ICHEP **2010**, 556 (2010) [arXiv:1103.1318 [hep-ph]]

The NLO revolution



The NLO revolution



BlackHat → Berger,Bern,Dixon,Febres Cordero,Forde,Ita,Kosower,Maître
HelacNLO → Bevilacqua,Czakon,Papadopoulos,Pittau,Worek
NJet → Badger,Biedermann,Uwer,Yundin
Rocket → Ellis,Melnikov,Zanderighi

MadGraph: →J. Alwall *et al.*, JHEP **1407** (2014) 079 [arXiv:1405.0301 [hep-ph]].

OpenLoops: → F. Cascioli, P. Maierhofer and S. Pozzorini, Phys. Rev. Lett. **108**, 111601 (2012) [arXiv:1111.5206 [hep-ph]].

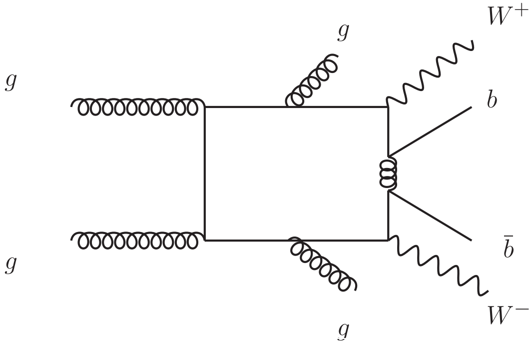
The NLO wishlist

Process ($V \in \{Z, W, \gamma\}$)	Status
1. $pp \to VV$ jet	WW jet completed by Dittmaier/Kallweit/Uwer; Campbell/Ellis/Zanderighi ZZ jet completed by Binoth/Gleisberg/Karg/Kauer/Sanguinetti WZ jet, $W\gamma$ jet completed by Campanario et al.
2. $pp \to$ Higgs+2 jets	NLO QCD to the gg channel completed by Campbell/Ellis/Zanderighi NLO QCD+EW to the VBF channel completed by Ciccolini/Denner/Dittmaier Interference QCD-EW in VBF channel
3. $pp \to V\,V\,V$	ZZZ completed by Lazopoulos/Melnikov/Petriello and WWZ by Hankele/Zeppenfeld see also Binoth/Ossola/Papadopoulos/Pittau VBFNLOmeanwhile also contains $WWW, ZZW, ZZZ, WW\gamma, ZZ\gamma, WZ\gamma, W\gamma\gamma, Z\gamma\gamma, \gamma\gamma\gamma, W\gamma\gamma j$
4. $pp \to t\bar{t}\,b\bar{b}$	relevant for $t\bar{t}H$, computed by Bredenstein/Denner/Dittmaier/Pozzorini and Bevilacqua/Czakon/Papadopoulos/Pittau/Worek
5. $pp \to V$+3 jets	W+3 jets calculated by the Blackhat/Sherpa and Rocket collaborations Z+3jets by Blackhat/Sherpa
6. $pp \to t\bar{t}$+2jets	relevant for $t\bar{t}H$, computed by Bevilacqua/Czakon/Papadopoulos/Worek
7. $pp \to VV\,b\bar{b}$,	Pozzorini et al.Bevilacqua et al.
8. $pp \to VV$+2jets	W^+W^++2jets, W^+W^-+2jets, relevant for VBF $H \to VV$ VBF contributions by (Bozzi/)Jäger/Oleari/Zeppenfeld Binoth et al.
9. $pp \to b\bar{b}b\bar{b}$	
10. $pp \to V+4$ jets	top pair production, various new physics signatures Blackhat/Sherpa: W+4jets, Z+4jets see also HEJfor $W+n$jets
11. $pp \to Wb\bar{b}j$	top, new physics signatures, Reina/Schutzmeier
12. $pp \to t\bar{t}t\bar{t}$	various new physics signatures, Bevilacqua/Worek
$pp \to W\gamma\gamma$ jet $pp \to 4/5$ jets	Campanario/Englert/Rauch/Zeppenfeld Blackhat+Sherpa/NJets

Closed

- NLO calculations requested by LHC experimenters
- List constructed in 2005
- Calculations completed 2012

→ G. Bevilacqua, M. Lupattelli, D. Stremmer and M. Worek, [arXiv:2212.04722 [hep-ph]].



NLO $2 \to 6$ ($2 \to 8$ including leptonic $W^\pm$ decays)

Towards higher precision:

NNLO and beyond

I have a dream ...

The two-loop frontier: $2 \rightarrow 3$

5-POINT 2-LOOP - MASSLESS: ALL FAMILIES

→ T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001 [erratum: Phys. Rev. Lett. **116** (2016) no.18, 189903] [arXiv:1511.05409 [hep-ph]].

→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].

→ D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang and S. Zoia, Phys. Rev. Lett. **123** (2019) no.4, 041603

→ D. Chicherin and V. Sotnikov, JHEP **20** (2020), 167

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, "Caravel: A C++ framework for the computation of multi-loop amplitudes with numerical unitarity," Comput. Phys. Commun. **267** (2021), 108069

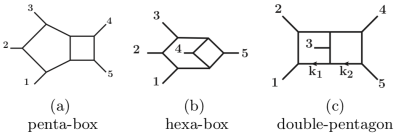
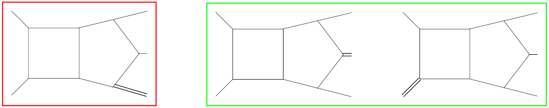


FIG. 1: Integral topologies for massless five-particle scattering at two loops.

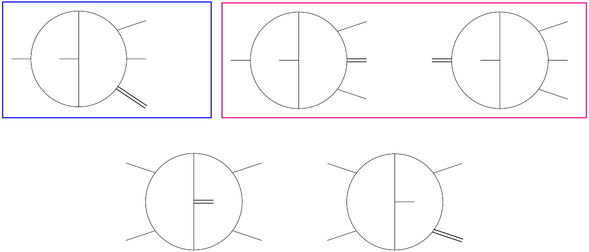
→ J. Henn, T. Peraro, Y. Xu and Y. Zhang, "A first look at the function space for planar two-loop six-particle Feynman integrals," JHEP **03** (2022), 056

5-POINT 2-LOOP - ONE LEG OFF-SHELL: ALL FAMILIES

→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].
→C. G. Papadopoulos and C. Wever, JHEP **2002** (2020) 112
→S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199
→ S. Abreu, H. Ita, B. Page and W. Tschernow, JHEP **03** (2022), 182 [arXiv:2107.14180 [hep-ph]].
→ A. Kardos, C. G. Papadopoulos, A. V. Smirnov, N. Syrrakos and C. Wever, [arXiv:2201.07509 [hep-ph]].
→ S. Abreu, D. Chicherin, H. Ita, B. Page, V. Sotnikov, W. Tschernow and S. Zoia, [arXiv:2306.15431 [hep-ph]].



The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

NNLO QCD: $pp \to \gamma\gamma\gamma + X$
leading-colour approximation for double-virtual

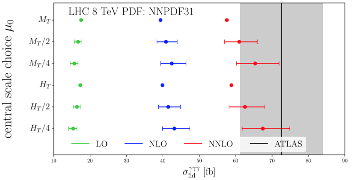


Figure 1. Predictions for the fiducial cross-section in LO (green), NLO (blue) and NNLO (red) QCD versus ATLAS data (black). Shown are predictions for six scale choices. The error bars on the theory predictions reflect scale variation only. For two of the scales only the central predictions are shown.

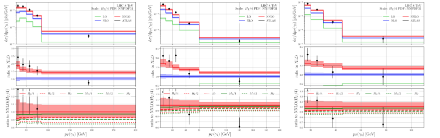


Figure 2. p_T distribution of the hardest photon γ_1 (left), γ_2 (center) and the softest one γ_3 (right). Top plot shows the absolute distribution at NNLO (red), NLO (blue) and LO (green) versus ATLAS data (black). Middle plot shows same distributions but normalized to the NLO. Bottom plot shows central NNLO predictions for 6 different scale choices (only the central scale is shown) with respect to the default choice $\mu_0 = H_T/4$. The bands represent the 7-point scale variations about the corresponding central scales.

→ H. A. Chawdhry, M. L. Czakon, A. Mitov and R. Poncelet, JHEP **2002** (2020) 057

NNLO QCD: $pp \to \gamma\gamma\gamma + X$
leading-colour approximation for double-virtual

fiducial setup for $pp \to \gamma\gamma\gamma + X$; used in the ATLAS 8 TeV analysis of Ref. [37]
$p_{T,\gamma_1} \geq 27\,\text{GeV}$, $p_{T,\gamma_2} \geq 22\,\text{GeV}$, $p_{T,\gamma_3} \geq 15\,\text{GeV}$, $0 \leq \lvert\eta_\gamma\rvert \leq 1.37$ or $1.56 \leq \lvert\eta_\gamma\rvert \leq 2.37$,
$\Delta R_{\gamma\gamma} \geq 0.45$, $m_{\gamma\gamma\gamma} \geq 50\,\text{GeV}$, Frixione isolation with $n = 1$, $\delta_0 = 0.4$, and $E_T^{\text{ref}} = 10\,\text{GeV}$.

Table 1: Definition of phase space cuts.

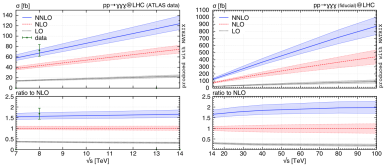


Figure 4: Fiducial cross sections for $pp \to \gamma\gamma\gamma + X$ as a function of the centre-of-mass energy at LO (black dotted), at NLO (red dashed), and at NNLO (blue, solid) The green data point at 8 TeV corresponds to the cross section measured by ATLAS in Ref. [37].

→ S. Kallweit, V. Sotnikov and M. Wiesemann, Phys. Lett. B **812** (2021) 136013

NNLO 2→3

NNLO QCD: $pp \to 3\text{jets} + X$

leading-colour approximation for double-virtual

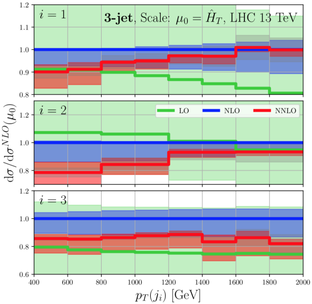
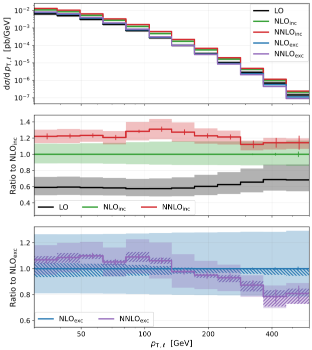


FIG. 1: The three panels show the ith leading jet transverse momentum $p_T(j_i)$ for $i = 1, 2, 3$ for the production of (at least) three jets. LO (green), NLO (blue) and NNLO (red) are shown for the central scale (solid line). 7-point scale variation is shown as a coloured band. The grey band corresponds to the uncertainty from Monte Carlo integration.

→ M. Czakon, A. Mitov and R. Poncelet, Phys. Rev. Lett. **127** (2021) no.15, 152001 [arXiv:2106.05331 [hep-ph]].

→X. Chen, T. Gehrmann, N. Glover, A. Huss and M. Marcoli, [arXiv:2203.13531 [hep-ph]].

NNLO QCD: $pp \to Wb\bar{b} + X$
leading-colour approximation for double-virtual



→ H. B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, Phys. Rev. D **106** (2022) no.7, 074016 [arXiv:2205.01687 [hep-ph]].

NNLO QCD: $pp \to \gamma j_1 j_2 + X$
Full-colour; sub-leading colour contributions negligible

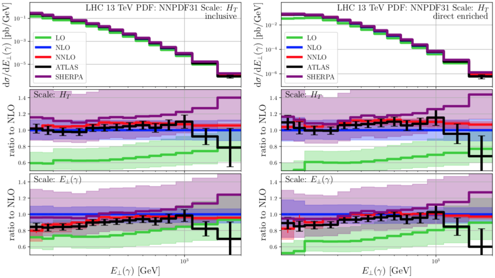


Figure 4. Differential cross sections w.r.t. the transverse energy of the photon $E_\perp(\gamma)$ in the *inclusive* (left plot) and *direct-enriched* (right plot) phase space at LO (green), NLO (blue) and NNLO (red) QCD compared to data (black) and SHERPA (purple) prediction provided by ATLAS[37]. The top panels show the absolute values for the H_T scale choice. The middle (bottom) panel shows the ratio to NLO QCD using the H_T ($E_\perp(\gamma)$) scale. The coloured bands show scale variation and the vertical coloured bars indicate statistical uncertainties.

→ S. Badger, M. Czakon, H. B. Hartanto, R. Moodie, T. Peraro, R. Poncelet and S. Zoia, [arXiv:2304.06682 [hep-ph]].

NNLO 2→3

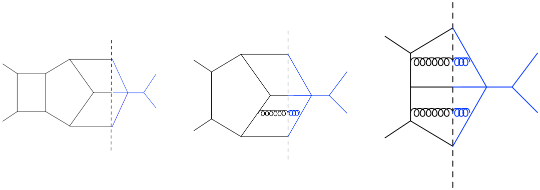
→ V. Sotnikov, [arXiv:2207.12295 [hep-ph]].

	Comment	Complete analytic results	Public numerical code	Cross sections
$pp \to jjj$	l.c.	Abreu et al.	Abreu et al.	Chen et al., Czakon et al.
$pp \to \gamma\gamma j$	l.c.*	Agarwal et al., Chawdhry et al.	Agarwal et al.	Chawdhry et al.
$pp \to \gamma\gamma\gamma$	l.c.*	Abreu et al., Chawdhry et al.	Abreu et al.	Chawdhry et al., Kallweit et al.
$pp \to \gamma\gamma j$		Agarwal et al.		
$gg \to \gamma\gamma g$	NLO loop induced	Badger et al.	Badger et al.	Badger et al.
$pp \to Wb\bar{b}$	l.c.*, on-shell W	Badger et al.		
$pp \to W(l\nu)b\bar{b}$	l.c.	Abreu et al., Hartanto et al.		Hartanto et al.
$pp \to W(l\nu)jj$	l.c.	Abreu et al.		
$pp \to Z(l\bar{l})jj$	l.c.*	Abreu et al.		
$pp \to W(l\nu)\gamma j$	l.c.*	Badger et al.		
$pp \to Hb\bar{b}$	l.c., b-quark Yukawa	Badger et al.		

Table 1: Known two-loop QCD corrections for five-point scattering processes at hardon colliders. "l.c." refers to the calculations in the leading-color approximation; "l.c.*" means that in addition non-planar l.c. contributions are omitted. All public codes employ PentagonFunctions++ Chicherin and Sotnikov, Chicherin et al. for numerical evaluation of special functions.

What do we need for an NNLO calculation ?

$$p_1, p_2 \rightarrow p_3, \ldots, p_{m+2}$$



What do we need for an NNLO calculation ?

$$\sigma_{NNLO} \rightarrow \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) \qquad VV$$

$$+ \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) \qquad RV$$

$$+ \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) \qquad RR$$

$RV + RR \rightarrow$ antenna-S, colorfull-NNLO, sector-improved residue subtraction, nested soft-collinear, local analytic sector subtraction, projection to born, q_T, N-jetiness

→A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
→ P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
→ M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
→S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
→ R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002
→ M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam and G. Zanderighi, Phys. Rev. Lett. **115**, no. 8, 082002 (2015)
→ F. Caola, K. Melnikov and R. Röntsch, Eur. Phys. J. C **77**, no. 4, 248 (2017)
→ L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, arXiv:1806.09570 [hep-ph].

Amplitude reduction

OPP AT TWO LOOPS

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\texttt{cut equations}: D_{i_1} = D_{i_2} = \ldots = D_{i_m} = 0$$

$$\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\}) \to spurious \oplus ISP - irreducible\ integrals$$

OPP AT TWO LOOPS

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

→ P. Mastrolia, T. Peraro and A. Primo, arXiv:1605.03157 [hep-ph].

→J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

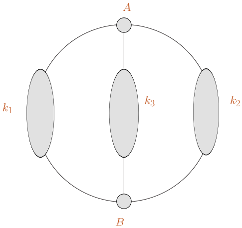
→ H. Ita, arXiv:1510.05626 [hep-th].

→ C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, Comput. Phys. Commun. **267** (2021), 108069

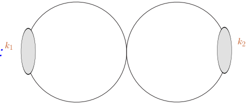
→ G. Bevilacqua, D. D. Canko, A. Kardos and C. G. Papadopoulos, J. Phys. Conf. Ser. **2105** (2021) no.5, 012010

1)Theta:



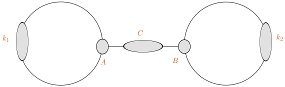
$\equiv \{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\}$,

2)Infinity:

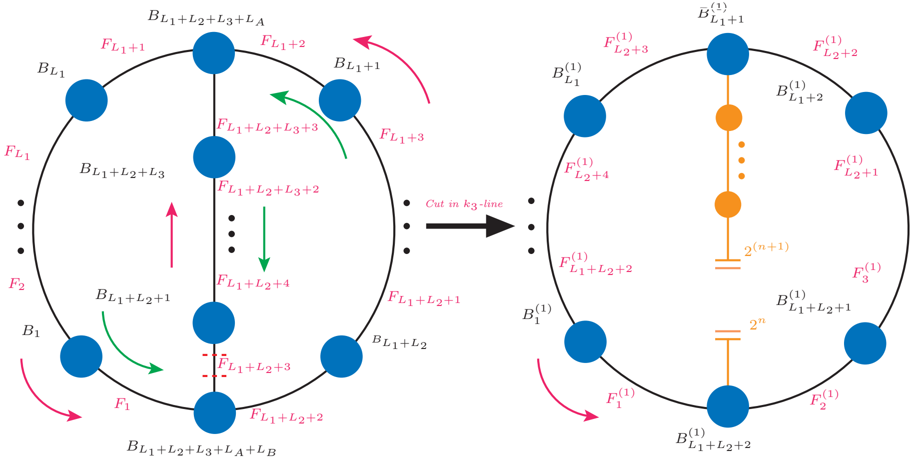


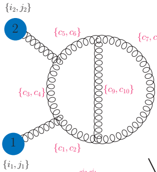
$\equiv \{\{k_1\},\{k_2\}\}$,

3)Dumbbell:

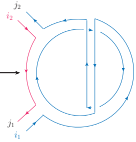


$\equiv \{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\}$.


$B_{L_1+L_2+L_3+L_A}$
F_{L_1+1}
F_{L_1+2}
B_{L_1}
B_{L_1+1}
$F_{L_1+L_2+L_3+3}$
F_{L_1}
F_{L_1+3}
$B_{L_1+L_2+L_3}$
$F_{L_1+L_2+L_3+2}$
Cut in k_3-line
F_2
$F_{L_1+L_2+4}$
$F_{L_1+L_2+1}$
$B_{L_1+L_2+1}$
B_1
$B_{L_1+L_2}$
$F_{L_1+L_2+3}$
F_1
$F_{L_1+L_2+2}$
$B_{L_1+L_2+L_3+L_A+L_B}$
$\bar{B}^{(1)}_{L_1+1}$
$F^{(1)}_{L_2+3}$
$F^{(1)}_{L_2+2}$
$B^{(1)}_{L_1}$
$B^{(1)}_{L_1+2}$
$F^{(1)}_{L_2+4}$
$F^{(1)}_{L_2+1}$
$2^{(n+1)}$
$F^{(1)}_{L_1+L_2+2}$
$F^{(1)}_3$
$B^{(1)}_{L_1+L_2+1}$
$B^{(1)}_1$
2^n
$F^{(1)}_1$
$F^{(1)}_2$
$B^{(1)}_{L_1+L_2+2}$

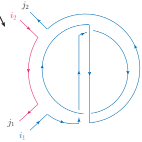


$\delta^{i_2}_{j_1}\delta^{i_1}_{j_2}$



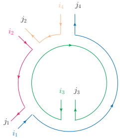
$\{c_1, c_2\} = \{i_1, i_1\}$
$\{c_3, c_4\} = \{i_2, i_1\}$
$\{c_5, c_6\} = \{i_1, i_1\}$
$\{c_7, c_8\} = \{i_1, i_1\}$
$\{c_9, c_{10}\} = \{i_1, i_1\}$

$C_F = -\delta^{i_1}_{c_1}\delta^{c_1}_{c_7}\delta^{c_7}_{c_{10}}\delta^{c_{10}}_{c_2}\delta^{c_2}_{c_4}\delta^{c_4}_{c_6}\delta^{c_6}_{c_8}\delta^{c_8}_{c_9}\delta^{c_9}_{c_5}\delta^{c_5}_{j_2}\delta^{i_2}_{c_3}\delta^{c_3}_{j_1}$



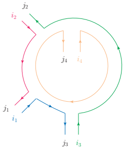
$\{c_1, c_2\} = \{i_1, i_1\}$
$\{c_3, c_4\} = \{i_2, i_1\}$
$\{c_5, c_6\} = \{i_1, i_1\}$
$\{c_7, c_8\} = \{i_1, i_1\}$
$\{c_9, c_{10}\} = \{i_1, i_1\}$

$C_F = -\delta^{i_1}_{c_1}\delta^{c_1}_{c_9}\delta^{c_9}_{c_8}\delta^{c_8}_{c_2}\delta^{c_2}_{c_4}\delta^{c_4}_{c_6}\delta^{c_6}_{c_{10}}\delta^{c_{10}}_{c_7}\delta^{c_7}_{c_5}\delta^{c_5}_{j_2}\delta^{i_2}_{c_3}\delta^{c_3}_{j_1}$

1) 

$\{c_1, c_2\} = \{i_1, i_3\}$
$\{c_3, c_4\} = \{i_2, i_3\}$
$\{c_5, c_6\} = \{i_4, i_3\}$
$\{c_7, c_8\} = \{i_1, i_3\}$

$\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}\delta^{i_4}_{j_2}$

2) 

$\{c_1, c_2\} = \{i_1, i_4\}$
$\{c_3, c_4\} = \{i_2, i_4\}$
$\{c_5, c_6\} = \{i_3, i_4\}$
$\{c_7, c_8\} = \{i_3, i_4\}$

$\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_4}$

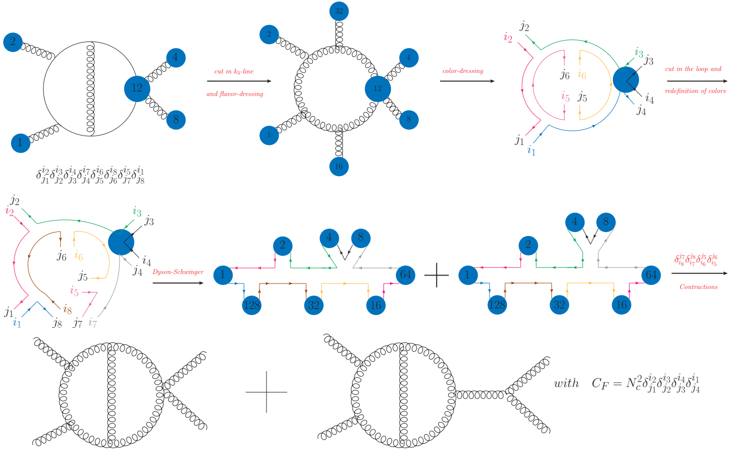


Figure: Schematic example of the construction of a two-loop contribution within HELAC-2LOOP.

```
INFO ===============================================
INFO COLOR           9 out of          24
INFO number of nums           0
INFO ===============================================
INFO COLOR          10 out of          24
INFO number of nums         332
INFO NUM           1  of          332          8
```

```
 INFO NUM          110  of          332          7
 INFO =====================================
INFO   4  80  35   9   1   1  16  35   5  64  35   7   0   0   0   0   1   2
INFO   4  12  35  10   1   1   4  35   3   8  35   4   0   0   0   0   1   1
INFO   4  92  35  11   1   2  12  35  10  80  35   9   0   0   0   0   1   1
INFO   5  92  35  11   2   2   4  35   3   8  35   4  80  35   9   0   1   5
INFO   4 124  35  12   1   1  32  35   6  92  35  11   0   0   0   0   1   2
INFO   4 126  35  13   1   1   2  35   2 124  35  12   0   0   0   0   1   1
INFO   4 254  35  14   1   1 128  35   8 126  35  13   0   0   0   0   1   2
INFO   6   1  12   1   2  12  35  35  35  35  35  35   0   0   0   0   5   9
```

FIGURE: Form of the contribution constructed in previous Figure, as is stored in the skeleton.

HELAC2LOOP: The algorithm

Process	Loops	*Loop-Flavors*	*Color*	*Skeleton Size*	*Timing*	*Numerators*
$gg \to gg$	2	$\{g, c, \bar{c}\}$	Leading	8.9 MB	15.017s	4560
$gg \to gg$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	110.6 MB	6m 54.574s	89392
$gg \to q\bar{q}$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	16.1 MB	3m 14.509s	13856
$gg \to ggg$	2	$\{g, c, \bar{c}\}$	Leading	300.0 MB	21m 42.609s	81480
$gg \to gg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	537.8 kB	2.386s	768
$gg \to ggg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	15.1 MB	8m 53.349s	11496
$gg \to gggg$	1	$\{g, c, \bar{c}\}$	Leading	394.0 MB	104m 14.95s	19680

Table: Table containing information for the skeleton of some QCD processes at one- and two-loop. These results have been obtained running 1-core in a personal laptop (i7 processor, 8-core, 25GB RAM).

1)$k_i \cdot k_j \rightarrow \#_1 = 3$, 2)$k_i \cdot p_j \rightarrow \#_2 = \min[4, n-1] \times 2$, 3)$k_i \cdot \eta_j \rightarrow \#_3 = 8 - \#_2$, with $\eta_i \perp p_j$.

$$k_i = \bar{k}_i + k_i^* \quad \text{with} \quad k_i \cdot k_j = \bar{k}_i \cdot \bar{k}_j + \mu_{ij}, \quad \text{and} \quad \mu_{ij} = k_i^* \cdot k_j^* .$$

$$N_I = \sum_{m=1}^{n_I} \left(\sum_{\mathbf{i_m}} c_{\mathbf{i_m}} \prod_{i \notin \mathbf{i_m}} D_i \right) \quad \text{with} \quad \mathbf{i_m} = \{i_1, \cdots, i_{m-1}, i_m\},$$

$$c_{\mathbf{i_m}} = \sum_{j=1} \tilde{c}_{\mathbf{i_m}}^{(j)}(\vec{s}, \varepsilon) \left(\bar{z}_1^{(\mathbf{i_m})} \right)^{\alpha_1^{(j)}} \cdots \left(\bar{z}_{n_{ir}}^{(\mathbf{i_m})} \right)^{\alpha_{n_{ir}}^{(j)}} ,$$

1) $(k_i \cdot \eta_j)^2 \rightarrow (k_i \cdot \eta_j)^2 - \dfrac{\mu_{ii}}{d-4}$

2) $(k_i \cdot \eta_j)^2 (k_{i'} \cdot \eta_j)^2 \rightarrow (k_i \cdot \eta_j)^2 (k_{i'} \cdot \eta_j)^2 - \dfrac{(k_i \cdot \eta_j)^2 \mu_{i'i'} + (k_{i'} \cdot \eta_j)^2 \mu_{ii} + 4(k_i \cdot \eta_j)(k_{i'} \cdot \eta_j)\mu_{ii'}}{2(d-4)}$.

HELAC2LOOP: The algorithm

1. Determination of the 4-dimensional part of $\tilde{c}^{(j)}_{\mathbf{i_m}}(\vec{s}, d)$ using values for the loop-momenta obtained from the cut equations $D_{i_1} = \cdots = D_{i_m} = 0$ in an OPP-like approach.
2. Determining the (polynomial) dependence on μ_{ij} terms ($\mathcal{R}_1$).
3. Determination of the missing ε-dimensional part of the numerator using two-loop rational terms ($\mathcal{R}_2$).

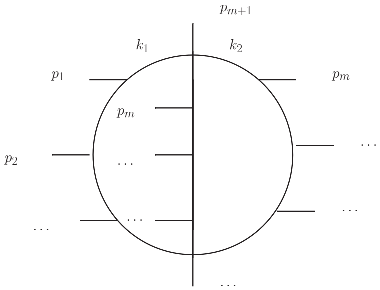
→S. Pozzorini, H. Zhang and M. F. Zoller, JHEP **05** (2020), 077

→J. N. Lang, S. Pozzorini, H. Zhang and M. F. Zoller, JHEP **10** (2020), 016

4. IBP reduction of the Feynman integrals resulted from ISP monomials to MIs.

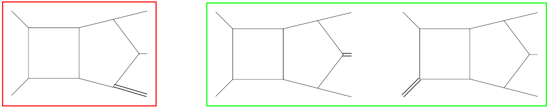
Feynman Integrals

Perturbative QCD at NNLO

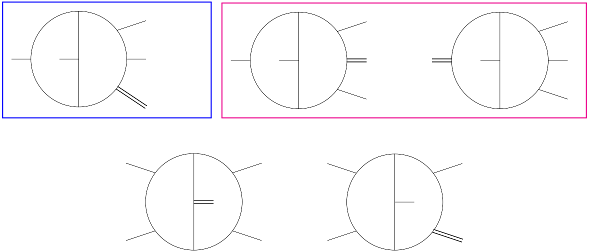


$$\frac{\mathcal{N}\left(k_1, k_2; \{p_i\}_{i=1}^{m+1}, \{\varepsilon\}\right)}{\left(k^2 - {M_0}^2\right)\left((k_1+p_1)^2 - {M_1}^2\right)\dots\left((k_1-k_2-p_{m+1})^2 - {M_j}^2\right)\dots\left({k_2}^2 - {M_l}^2\right)}$$

The SDE approach



The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

PENTABOX - ONE LEG OFF-SHELL: P1

→J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601

→S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117

→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

- Also from direct differentiation of MI wrt to x (Fuchsian).

$$\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \vec{g}$$

- ℓ_b, are independent of x, some depending only on the reduced invariants, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. M_b are independent of the invariants.
- number of letters smaller than in AIMPTZ representation
- Main contribution for us from AIMPTZ: the canonical basis (+ numerics)

Pentabox - one leg off-shell: P1

→J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601

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→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

$$\frac{d\log\left(W_a\right)}{dx}$$

- Also from direct differentiation of MI wrt to x (Fuchsian).

$$\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \vec{g}$$

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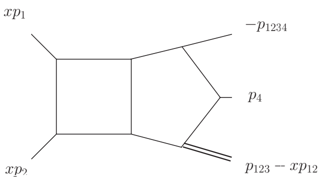
PENTABOX - ONE LEG OFF-SHELL: P1



$$q_1 \to p_{123} - xp_{12},\ q_2 \to p_4,\ q_3 \to -p_{1234},\ q_4 \to xp_1$$

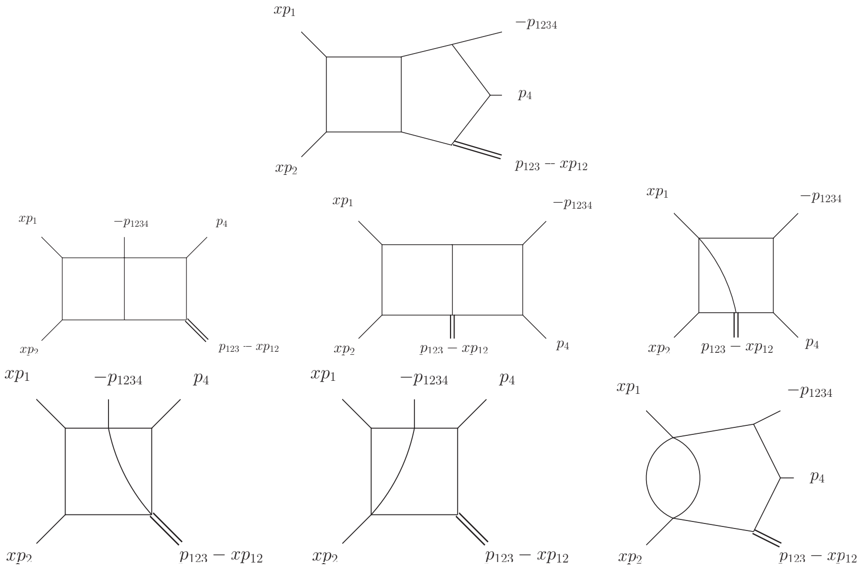
SDE parametrisation: n off-shell legs $\to$ $n-1$ off-shell legs $+$ the x variable.

→ C. G. Papadopoulos, "Simplified differential equations approach for Master Integrals," JHEP **1407** (2014) 088

- $p_i,\ i = 1\ldots5$, satisfy $\sum_1^5 p_i = 0$, with $p_i^2 = 0,\ i = 1\ldots5,\ p_{i\ldots j} := p_i + \ldots + p_j$. The set of independent invariants: $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$.

$$q_1^2 = (1-x)(S_{45} - S_{12}x),\ s_{12} = (S_{34} - S_{12}(1-x))\,x,\ s_{23} = S_{45},\ s_{34} = S_{51}x,$$
$$s_{45} = S_{12}x^2,\ s_{15} = S_{45} + (S_{23} - S_{45})x$$

Pentabox - one leg off-shell: P1



→ J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1405** (2014) 090
→ T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs, JHEP **06** (2014), 032
→ F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1409** (2014) 043
→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072
→ T. Gehrmann, A. von Manteuffel and L. Tancredi, JHEP **09** (2015), 128

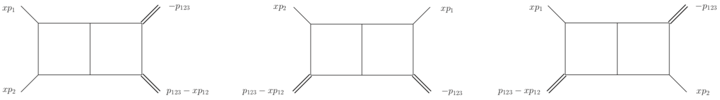


Figure 3. The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

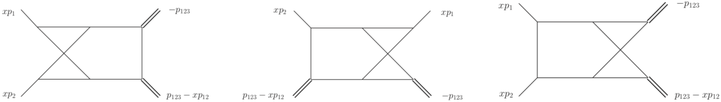


Figure 4. The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

As well as planar and nonplanar double box with one off-shell leg expressed in UT basis.

Pentabox - one leg off-shell: P1-3

$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_a \frac{1}{x - \ell_a} \mathbf{M}_a \mathbf{g}$$

Pentabox - one leg off-shell: P1-3

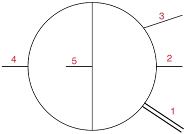
$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_a \frac{1}{x - \ell_a} \mathbf{M}_a \mathbf{g}$$

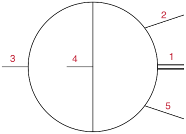
$$\begin{aligned}
\mathbf{g} = \epsilon^0 \mathbf{b}_0^{(0)} &+ \epsilon \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\
&+ \epsilon^2 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) \\
&+ \epsilon^3 \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)} \right) \\
&+ \epsilon^4 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)} \right. \\
&\left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)} \right) + \ldots
\end{aligned}$$

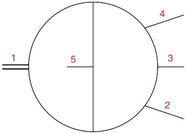
$$\mathcal{G}_{ab\ldots} := \mathcal{G}(\ell_a, \ell_b, \ldots; x)$$

→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. Results.txt

HEXABOX - ONE LEG OFF-SHELL







$$r_1 = \sqrt{\lambda(p_{1s},\ s_{23},\ s_{45})}$$

$$r_2 = \sqrt{\lambda(p_{1s},\ s_{24},\ s_{35})}$$

$$r_3 = \sqrt{\lambda(p_{1s},\ s_{25},\ s_{34})}$$

$$r_4 = \sqrt{\det \mathbb{G}(q_1,\ q_2,\ q_3,\ q_4)}$$

$$r_5 = \sqrt{\Sigma_5^{(1)}}$$

$$r_6 = \sqrt{\Sigma_5^{(2)}}$$

- For topology N_1, the square roots r_1 and r_4 appear in its alphabet and are rationalized.

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{l_{max}} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g}$$

  $l_{max} = 21$ from 39 letters in the original alphabet

- For topologies N_2 and N_3, the square roots appearing are $\{r_1, \ r_2, \ r_4, \ r_5\}$ and $\{r_1, \ r_3, \ r_4, \ r_6\}$ not *simultaneous* rationalisation possible !

The more general form of the SDE takes the form:

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{a=1}^{l_{max}} \frac{dL_a}{dx} \mathbf{M}_a \right) \mathbf{g}$$

where most of the L_a are simple rational functions of x , whereas the rest are algebraic functions of x involving the non-rationalisable square roots.

- One-dimensional integration based on weight-2 functions

→ S. Caron-Huot and J. M. Henn, JHEP **06** (2014), 114

For instance element 11 of N_2 is given as

$$g_{11}^{(2)} = 8\left(2\mathcal{G}(0,-y)\left(\mathcal{G}\left(1,y\right)-\mathcal{G}\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)\right)+2\mathcal{G}\left(0,\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)-\mathcal{G}\left(1,y\right)\log\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}}\right)\right.$$
$$\left.+\log\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}}\right)\mathcal{G}\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)-2\mathcal{G}\left(0,1,y\right)\right)$$

where the new parametrization of the external momenta is given by

$$q_1 \to \tilde{p}_{123} - y\tilde{p}_{12},\; q_2 \to y\tilde{p}_2,\; q_3 \to -\tilde{p}_{1234},\; q_4 \to y\tilde{p}_1$$

with the new momenta $\tilde{p}_i,\; i = 1\ldots5$ satisfying as usual, $\sum_1^5 \tilde{p}_i = 0,\; \tilde{p}_i^2 = 0,\; i = 1\ldots5$, with $\tilde{p}_{i\ldots j} := \tilde{p}_i + \ldots + \tilde{p}_j$. The set of independent invariants is given by $\{\tilde{S}_{12}, \tilde{S}_{23}, \tilde{S}_{34}, \tilde{S}_{45}, \tilde{S}_{51}, y\}$, with $\tilde{S}_{ij} := (\tilde{p}_i + \tilde{p}_j)^2$. The explicit mapping between the two sets of invariants is given by

$$q_1^2 = (1-y)(\tilde{S}_{45} - \tilde{S}_{12}y),\; s_{12} = \tilde{S}_{45}(1-y) + \tilde{S}_{23}y,\; s_{23} = -y\left(\tilde{S}_{12} - \tilde{S}_{34} + \tilde{S}_{51}\right),$$
$$s_{34} = \tilde{S}_{51}y, s_{45} = y\left(\tilde{S}_{23} - \tilde{S}_{45} - \tilde{S}_{51}\right),\; s_{15} = y\left(\tilde{S}_{34} - \tilde{S}_{12}(1-y)\right).$$

- By identifying $f_- = y$ and $f_+ = y\frac{\tilde{S}_{12}}{\tilde{S}_{45}}$, which are given as

$$f_\pm = \frac{S_{45} + x\left(-S_{23} - S_{34} + 2S_{51} + S_{12}x\right) \pm r_2}{2\left(S_{12} - S_{34} + S_{51}\right)x}$$

we can write the DE for this element in the simple and compact form

$$\frac{d}{dx}g_{11}^{(2)} = -8\left(\mathrm{dlog}\left(\frac{f_+ - 1}{f_- - 1}\right)\log\left(f_- f_+\right) - \mathrm{dlog}\left(\frac{f_+}{f_-}\right)\log\left(\left(f_- - 1\right)\left(f_+ - 1\right)\right)\right).$$

The form of the DE makes the determination of the ansatz rather straightforward, with the result

$$g_{11}^{(2)} = -8\Bigg(-\log(f_- f_+)\Big(\mathcal{G}(1,f_-) - \mathcal{G}(1,f_+)\Big) + 2\mathcal{G}(0,1,f_-) - 2\mathcal{G}(0,1,f_+)\Bigg).$$

- Concerning the other non-rationalisable square root in the family N_2, r_5, it also appears for the first time at weight 2 in the basis element 73 only , which is one of the new integrals to be calculated.

$$g_{73}^{(2)} = 16\log\left(f_- f_+\right)\Big(\mathcal{G}(1,f_-) - \mathcal{G}(1,f_+)\Big) - 32\Big(\mathcal{G}(0,1,f_-) - \mathcal{G}(0,1,f_+)\Big)$$

with

$$f_\pm = \frac{S_{45}\left(2S_{12}x - S_{34}x + S_{51}\right) + x\left(S_{23}S_{34} - S_{12}S_{23} + xS_{12}S_{51}\right) \pm r_5}{2S_{45}\left(S_{12} - S_{34} + S_{51}\right)}$$

Weight 3:
The differential equation can be written in the form:

$$\partial_x g_I^{(3)} = \sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)}$$

Since the lower limit of integration corresponds to $x = 0$, we need to subtract the appropriate term so that the integral is explicitly finite. This is achieved as follows:

$$\partial_x g_I^{(3)} = \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} + \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right)$$

where $g_{I,0}^{(2)}$ are obtained by expanding $g_I^{(2)}$ around $x = 0$ and keeping terms up to order $\mathcal{O}\left(\log(x)^2\right)$, and $l_a \in \mathbb{Q}$ are defined through

$$\partial_x \log L_a = \frac{l_a}{x} + \mathcal{O}\left(x^0\right).$$

The DE can now be integrated from $x = 0$ to $x = \bar{x}$, and the result is given by

$$g_I^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right)$$

with $b_I^{(3)}$ being the boundary terms at $\mathcal{O}(\epsilon^3)$ and

$$g_{I,\mathcal{G}}^{(3)} = \int_0^{\bar{x}} \mathrm{d}x \sum_a \frac{l_a}{x} \sum_J c_{LJ}^a g_{J,0}^{(2)} \bigg|_{\mathcal{G}}$$

with the subscript $\mathcal{G}$, indicating that the integral is represented in terms of GPLs , following the convention

$$\int_0^{\bar{x}} dx \frac{1}{x} \mathcal{G}\left(\underbrace{0,\ldots 0}_{n}; x\right) = \mathcal{G}\left(\underbrace{0,\ldots 0}_{n+1}; \bar{x}\right).$$

Alternative for the analytical aficionados (AA): work out *linear letters* →Goncharov MPL

→For instance, N_2 element 11 known at w=3 in terms of y, as well as many other

HEXABOX - ONE LEG OFF-SHELL: INTEGRAL REP.

Weight 4:
At weight 4, the differential equation can be written in the form:

$$\partial_x g_I^{(4)} = \sum_a \Big(\partial_x \log L_a\Big) \sum_J c_{IJ}^a g_J^{(3)}$$

which after doubly-subtracting, in order to obtain integrals that are explicitly finite , is written as

$$\partial_x g_I^{(4)} = \sum_a \partial_x(\log L_a - LL_a) \sum_J c_{IJ}^a g_J^{(3)} + \sum_a \partial_x(LL_a) \sum_J c_{IJ}^a (g_J^{(3)} - g_{J,0}^{(3)}) + \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(3)}$$

where LL_a are obtained by expanding $\log(L_a)$ around $x=0$ and keeping terms up to order $\mathcal{O}\Big(\log(x)\Big)$, and

$$g_{I,0}^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)}.$$

HEXABOX - ONE LEG OFF-SHELL: INTEGRAL REP.

Now, by integrating by parts we can write the final result as follows:

$$
\begin{aligned}
g_I^{(4)} =& g_{I,\mathcal{G}}^{(4)} + b_I^{(4)} + \left(\sum_a \log L_a \sum_J c_{IJ}^a g_J^{(3)} \right) - \left(\sum_a LL_a \sum_J c_{IJ}^a g_{J,0}^{(3)} \right) \\
&- \int_0^{\bar{x}} \mathrm{d}x \sum_a (\log L_a - LL_a) \sum_J c_{IJ}^a \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)} \\
&- \int_0^{\bar{x}} \mathrm{d}x \sum_a \log L_a \sum_J c_{IJ}^a \left(\sum_b (\partial_x \log L_b) \sum_K c_{JK}^b g_K^{(2)} - \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)} \right)
\end{aligned}
$$

with a, b running over the set of contributing letters, I, J, K running over the set of basis elements, $b_I^{(4)}$ being the boundary terms at $\mathcal{O}(\epsilon^4)$ and

$$
g_{I,\mathcal{G}}^{(4)} = \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(3)} \right) \Bigg|_{\mathcal{G}}
$$

where the subscript $\mathcal{G}$ indicates that the integral is represented in terms of GPLs.
analytic continuation $\rightarrow$ applying fibration on $b_I^{(1\ldots4)}$ and g up to weight two

HEXABOX - ONE LEG OFF-SHELL: BOUNDARY TERMS

- The pure basis elements can be written in general as follows:

$$g = Ce^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{P\left(\{D_i\}, \{S_{ij}, x\}\right)}{\prod\limits_{i \in \tilde{S}} D_i^{a_i}} \tag{1}$$

  where D_i, $i = 1 \ldots 11$, represent the inverse scalar propagators, $\tilde{S}$ the set of indices corresponding to a given sector, S_{ij}, x the kinematic invariants, P is a polynomial, a_i are positive integers and C a factor depending on S_{ij}, x.

- This form is usually decomposed in terms of FI, F_i,

$$g = C \sum c_i\left(\{S_{ij}, x\}\right) F_i$$

  with c_i being polynomials in S_{ij}, x.

- An alternative approach, would be to build-up the Feynman parameter representation for the whole basis element, by considering the integral as a tensor integral in its Feynman parameter representation.

→ J. Gluza, K. Kajda, T. Riemann and V. Yundin, Eur. Phys. J. C **71** (2011), 1516 [arXiv:1010.1667 [hep-ph]].

→ S. C. Borowka, [arXiv:1410.7939 [hep-ph]].

Then, by using the expansion by regions approach

→ B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012), 2139 [arXiv:1206.0546 [hep-ph]].

→ A. V. Smirnov, Comput. Phys. Commun. **204** (2016), 189-199 [arXiv:1511.03614 [hep-ph]].

$$b = \sum_I N_I \int \prod_{i \in S_I} dx_i \, U_I^{a_I} F_I^{b_i} \Pi_I$$

where I runs over the set of contributing regions, U_I and F_I are the limits of the usual Symanzik polynomials, Π_I is a polynomial in the Feynman parameters, x_i, and the kinematic invariants S_{ij}, and S_I the subset of surviving Feynman parameters in the limit.

- In this way a significant reduction of the number of regions to be calculated is achieved, namely from 208 to 9. Notice that in contrast to the approach described in the previous paragraphs, only the regions $x^{-2\epsilon}$ and $x^{-4\epsilon}$ contribute to the final result, making thus the evaluation of the region-integrals simpler.
- Moreover, this approach overpasses the need for an IBP reduction of the basis elements in terms of MI.

HEXABOX - ONE LEG OFF-SHELL: PERFORMANCE

As a proof of concept, we have implemented the final formulae in `Mathematica`. We use `NIntegrate` to perform the one-dimensional integrals, after expressing all weight-2 functions in terms of classical polylogarithms following reference

→ H. Frellesvig, D. Tommasini and C. Wever, JHEP **03** (2016), 189 [arXiv:1601.02649 [hep-ph]].

- The user can easily assess the performance of this straightforward implementation by running the provided codes and look at the minimum number of digits in agreement with the high-precision results from Abreu et. al, as well as at the number of integrand evaluations performed by `NIntegrate`.
- Notice that the integrand expressions involve logarithms and classical polylogarithms Li_2 that are evaluated using very little CPU time.
- The parts of the formulae that can be represented in terms of GPLs up to weight four, as well as the results for the N_1 family, for which we have all basis elements in terms of GPLs up to weight four, are evaluated with `GiNaC`, as implemented in `PolyLogTools`.

→ J. Vollinga, Nucl. Instrum. Meth. A **559** (2006), 282-284 [arXiv:hep-ph/0510057 [hep-ph]].

→ C. Duhr and F. Dulat, JHEP **08** (2019), 135 [arXiv:1904.07279 [hep-th]].

HEXABOX - ONE LEG OFF-SHELL: PERFORMANCE

- In the current implementation we use the default parameters for `GiNaC` and the default parameters for `NIntegrate` with the exception of `WorkingPrecision` and `PrecisionGoal`, in order to obtain reasonable results within reasonable time, taking into account that the provided implementation serves merely as a demonstration of the correctness of our representations.
- For the Euclidean point the precision is typically of the order of 32 digits, which is compatible with `GiNaC` setup.
- For the physical point, the typical precision is of the order of 25 digits, which is compatible with the expected one taking into account the numerical value of the infinitesimal imaginary part assigned to the kinematical invariants.

The loop-by-loop approach

→In collaboration with Roberto Pittau

Loop by loop

$$\bar{u}_L(p) = \langle p \qquad u_L(p) = p] \qquad \bar{u}_R(p) = [p \qquad u_R(p) = p\rangle$$

$$\ell_3{}^\mu = \langle p_1 | \ \gamma^\mu | \ p_2] = \bar{u}_L(p_1)\,\gamma^\mu u_L(p_2) = \bar{u}(p_1)\,\gamma^\mu \omega_- u(p_2)$$

$$\ell_4{}^\mu = \langle p_2 | \ \gamma^\mu | \ p_1] = \bar{u}_L(p_2)\,\gamma^\mu u_L(p_1) = \bar{u}(p_2)\,\gamma^\mu \omega_- u(p_1)$$

$$\omega_- = \frac{1-\gamma_5}{2}$$

$$k_2 \cdot \ell_3 k_2 \cdot \ell_4 = \bar{u}(p_1)\,\not{k}_2 \omega_- u(p_2)\,\bar{u}(p_2)\,\not{k}_2 \omega_- u(p_1) = \mathrm{Tr}\left(\not{p}_1 \not{k}_2 \omega_- \not{p}_2 \not{k}_2 \omega_-\right) = \mathrm{Tr}\left(\not{p}_1 \not{k}_2 \not{p}_2 \not{k}_2 \omega_-\right)$$

$$k_2 \cdot \ell_3 k_2 \cdot \ell_4 = \frac{1}{2}\mathrm{Tr}\left(\not{p}_1 \not{k}_2 \not{p}_2 \not{k}_2\right) - \frac{1}{2}\mathrm{Tr}\left(\not{p}_1 \not{k}_2 \not{p}_2 \not{k}_2 \gamma_5\right) = 2\left(2 p_1 \cdot k_2 \ p_2 \cdot k_2 - p_1 \cdot p_2 \ k_2{}^2\right)$$

$$G(k_2, p_1, p_2) = \begin{pmatrix} k_2{}^2 & k_2 \cdot p_1 & k_2 \cdot p_2 \\ k_2 \cdot p_1 & 0 & p_1 \cdot p_2 \\ k_2 \cdot p_2 & p_1 \cdot p_2 & 0 \end{pmatrix}$$

$$G = \det\left(G(k_2, p_1, p_2)\right) = p_1 \cdot p_2 \left(2 p_1 \cdot k_2 \ p_2 \cdot k_2 - p_1 \cdot p_2 \ k_2{}^2\right)$$

$$k_2 \cdot \ell_3 k_2 \cdot \ell_4 = 2\frac{G}{p_1 \cdot p_2}$$

$$\mathcal{N}_1 \equiv k_1 \cdot p_3 = d_{1234} + \tilde{d}_{1234}\left(\ell_3 \cdot k_1 \ell_4 \cdot k_2 - \ell_4 \cdot k_1 \ell_3 \cdot k_2\right) + c_{123} D_4 + c_{124} D_3 + c_{134} D_2 + c_{234} D_1$$

$$d_{1234} = -p_1 \cdot p_3 - \frac{1}{2}(k_2 - p_1)^2 \frac{p_1 \cdot p_3 p_2 \cdot k_2 + p_2 \cdot p_3 p_1 \cdot k_2 - p_1 \cdot p_2 p_3 \cdot k_2}{2 p_1 \cdot k_2 p_2 \cdot k_2 - p_1 \cdot p_2 k_2{}^2}$$

$$d_{1234} = -p_1 \cdot p_3 - \frac{1}{4}(k_2 - p_1)^2 \left(\frac{\ell_3 \cdot p_3}{\ell_3 \cdot k_2} + \frac{\ell_4 \cdot p_3}{\ell_4 \cdot k_2}\right)$$

$$\tilde{d}_{1234} = \frac{1}{8 p_1 \cdot p_2}\left(\frac{\ell_3 \cdot p_3}{\ell_3 \cdot k_2} - \frac{\ell_4 \cdot p_3}{\ell_4 \cdot k_2}\right)$$

$$c_{123} = \frac{1}{4}\left(\frac{\ell_3 \cdot p_3}{\ell_3 \cdot k_2} + \frac{\ell_4 \cdot p_3}{\ell_4 \cdot k_2}\right)$$

$$c_{124} = \frac{1}{2 p_1 \cdot p_2}\left(p_1 \cdot p_3 - \frac{1}{2} p_1 \cdot k_2 \left(\frac{\ell_3 \cdot p_3}{\ell_3 \cdot k_2} + \frac{\ell_4 \cdot p_3}{\ell_4 \cdot k_2}\right)\right)$$

$$c_{134} = \frac{1}{2 p_1 \cdot p_2}\left(p_2 \cdot p_3 - p_1 \cdot p_3 + \frac{1}{2}\left(p_1 \cdot k_2 - p_2 \cdot k_2\right)\left(\frac{\ell_3 \cdot p_3}{\ell_3 \cdot k_2} + \frac{\ell_4 \cdot p_3}{\ell_4 \cdot k_2}\right)\right)$$

$$c_{234} = -\frac{1}{2 p_1 \cdot p_2}\left(p_2 \cdot p_3 + \frac{1}{2}\left(p_1 \cdot p_2 - p_2 \cdot k_2\right)\left(\frac{\ell_3 \cdot p_3}{\ell_3 \cdot k_2} + \frac{\ell_4 \cdot p_3}{\ell_4 \cdot k_2}\right)\right)$$

$$\mathcal{N} = -p_1 \cdot p_3 + k_2 \cdot p_1 \, \frac{p_1 \cdot p_3 p_2 \cdot k_2 + p_2 \cdot p_3 p_1 \cdot k_2 - p_1 \cdot p_2 p_3 \cdot k_2}{2 p_1 \cdot k_2 \; p_2 \cdot k_2 - p_1 \cdot p_2 \; {k_2}^2}$$

$$\mathcal{N} = -p_1 \cdot p_3 + \frac{D_8}{D_{10}} \left(p_1 \cdot p_3 \frac{1}{2} \left(D_5 - 2D_8 - D_7 + {p_{12}}^2 \right) + p_2 \cdot p_3 D_8 - p_1 \cdot p_2 \frac{1}{2} \left(D_7 - D_6 - {p_{12}}^2 \right) \right)$$

$$\mathcal{N} = -p_1 \cdot p_3 + \frac{1}{2} p_1 \cdot p_3 \frac{D_8 \left(D_5 - 2D_8 - D_7 + {p_{12}}^2 \right)}{D_{10}} + p_2 \cdot p_3 \frac{{D_8}^2}{D_{10}} - \frac{1}{2} p_1 \cdot p_2 \frac{D_8 \left(D_7 - D_6 - {p_{12}}^2 \right)}{D_{10}}$$

$$\frac{D_8}{D_{10}}, \; \frac{D_8 D_5}{D_{10}}, \; \frac{D_8 D_6}{D_{10}}, \; \frac{D_8 D_7}{D_{10}}, \; \frac{{D_8}^2}{D_{10}}$$

$$
\begin{aligned}
\mathcal{N} &= -p_1 \cdot p_3 F[1,1,1,1,1,1,1,0,0,0] \\
&+\tfrac{1}{2} p_1 \cdot p_3 F[1,1,1,1,0,1,1,-1,0,1] \\
&+\tfrac{1}{2} p_1 \cdot p_3 (-2) F[1,1,1,1,1,1,1,-2,0,1] \\
&+\tfrac{1}{2} p_1 \cdot p_3 (-1) F[1,1,1,1,1,1,0,-1,0,1] \\
&+\tfrac{1}{2} p_1 \cdot p_3 \left({p_{12}}^2\right) F[1,1,1,1,1,1,1,-1,0,1] \\
&+p_2 \cdot p_3 F[1,1,1,1,1,1,1,-2,0,1] \\
&-\tfrac{1}{2} p_1 \cdot p_2 F[1,1,1,1,1,1,0,-1,0,1] \\
&-\tfrac{1}{2} p_1 \cdot p_2 (-1) F[1,1,1,1,1,0,1,-1,0,1] \\
&-\tfrac{1}{2} p_1 \cdot p_2 \left(-{p_{12}}^2\right) F[1,1,1,1,1,1,1,-1,0,1]
\end{aligned}
$$

→ Laporta algorithm reduces this to the known result.

→by special use of `FIRE6` thanks to A. V. Smirnov

Loop by loop

→ For higher-rank tensors

$$g = -s{k_2}^2 + 4k_2 \cdot p_1 k_2 \cdot p_2$$

$$g_1 = s{k_2}^2 - (k_2 \cdot p_1 + k_2 \cdot p_2)^2$$

$$g_2 = -\frac{s}{2} + k_2 \cdot p_2$$

$$g_3 = k_2 \cdot p_1$$

$$F^{(9)},\ F^{(10)}[g],\ F^{(10)}[g_1],\ F^{(10)}[g_2],\ F^{(10)}[g_3],\ F^{(11)}[g, g_1],\ F^{(11)}[g, g_2],\ F^{(11)}[g, g_3]$$

→ k_1-reduction increasing the rank of k_2 terms → k_2-reduction?

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→ A. Kardos, C. G. Papadopoulos, A. V. Smirnov, N. Syrrakos and C. Wever, JHEP **05** (2022), 033

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→ S. Abreu, H. Ita, B. Page and W. Tschernow, JHEP **03** (2022), 182

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  - SDE@1-loop → N. Syrrakos, "One-loop Feynman integrals for 2 → 3 scattering involving many scales including internal masses," JHEP **10** (2021), 041 [arXiv:2107.02106 [hep-ph]].
  - SDE@3-loop → D. D. Canko and N. Syrrakos, "Planar three-loop master integrals for 2 → 2 processes with one external massive particle," [arXiv:2112.14275 [hep-ph]].
  - UT basis determination → more criteria as experience dictates

    → H. Frellesvig and C. G. Papadopoulos, JHEP **04** (2017), 083

    → J. Henn, B. Mistlberger, V. A. Smirnov and P. Wasser, JHEP **04** (2020), 167

    → P. Wasser, "Scattering Amplitudes and Logarithmic Differential Forms,"

    → C. Dlapa, X. Li and Y. Zhang, JHEP **07** (2021), 227
  - Boundary terms determination → for UT basis elements
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Thank you for your attention !

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