Two-Loop Amplitude Reduction using ***HELAC***

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Collaboration with: C. Papadopoulos, & G. Bevilacqua.

Based on: J.Phys.Conf.Ser. 2105 (2021) 5, 012010, PoS CORFU2021 (2022) 010
& ongoing work

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Current status @ NNLO (QCD)

- Accurate data (HL-LHC/future colliders) → High-precision theoretical predictions!

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- At NNLO the hard-part cross-section, $\hat{\sigma}_{ab \to X}$, receives contributions from

$$d\hat{\sigma}^{NNLO}_{ab \to X} \sim |\mathcal{A}_{tree}|^2 + \alpha_S \left(2\,\text{Re}\left[\mathcal{A}_{tree}\mathcal{A}^*_{loop}\right] + |\mathcal{A}_{+1up}|^2 \right)$$
$$+ \alpha_S^2 \left(|\mathcal{A}_{loop}|^2 + 2\,\text{Re}\left[\mathcal{A}_{tree}\mathcal{A}^*_{2-loop}\right] + |\mathcal{A}_{+2up}|^2 + 2\,\text{Re}\left[\mathcal{A}_{loop+1up}\mathcal{A}^*_{+1up}\right] \right)$$

Final result finite using Renormalization and dimensional Regularization!

$2 \rightarrow 3$ Scattering Amplitudes results in the last years:

Gulio's, Michał's, and Simone's talks

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- $u\bar{d} \to W^{+} + b\bar{b}$ (LC) S. Badger, H. Hartanto and S. Zoia [Phys.Rev.Lett. 127 (2021) 1, 012001].
- $W + 4$ *partons* (LC) H. Hartanto, S. Badger, C. Brønnum-Hansen and T. Peraro [JHEP 09 (2019) 119], S. Abreu, F. Febres Cordero, H. Ita, M. Klinkert, B. Page and V. Sotnikov [JHEP 04 (2022) 042].
- $gg \to g\gamma\gamma$ (LC) S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H. Hartanto, J. Henn, M. Marcoli, R. Moodie, T. Peraro and S. Zoia [JHEP 11 (2021) 083].
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- $pp \to 3j$ (LC) S. Abreu, F. Febres Cordero, H. Ita, B. Page, and V. Sotnikov [JHEP 07 (2021) 095].
- 5 *partons* (LC) S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov [JHEP 05 (2019) 084], S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov [JHEP 11 (2018) 116].
- $gg \to ggg$ (LC) S. Abreu, J. Dormans, F. Febres Cordero, H. Ita and B. Page [Phys.Rev.Lett. 122 (2019) 8, 082002], S. Abreu, F. Febres Cordero, H. Ita, B. Page and M. Zeng [Phys.Rev.D 97 (2018) 11, 116014], S. Badger, H. Frellesvig, Y. Zhang [JHEP 12 (2013) 045], S. Badger, C. Brønnum-Hansen, H. Hartanto and T. Peraro [JHEP 01 (2019) 186].

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- $q\bar{q} \to gg\gamma$ and $q\bar{q} \to Q\bar{Q}\gamma$ (FC) S. Badger, M. Czakon, H. Hartanto, R. Moodie, T. Peraro, R. Poncelet, S. Zoia [2304.06682 [hep-ph]].
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- $gg \to ggg$ (FC+all-plus helicity) S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J.M. Henn, T. Peraro, P. Wasser, Y. Zhang and S. Zoia [Phys.Rev.Lett. 123 (2019) 7, 071601].

Apologize for missing references herein and from here on!

Workflow for 2-loop scattering amplitude computations

1) Construction of the Amplitude for the process at hand:

- *Sum up Feynman graphs (QGRAF, FeynArts)* P. Nogueira [J.Comp.Phys. 105 (1993) 279-289] T. Hahn [hep-ph/0012260]
- *Dyson-Schwinger recursion*
- *Hybrid approach* ⟵ *HELAC-2LOOP*

$$\mathcal{A}_{2-loop} = \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d} \mu^{2(d-4)}} A_{2-loop} = \sum_{I \subseteq T} \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d} \mu^{2(d-4)}} \frac{N_I(k_1, k_2, p_1, \ldots, p_{n-1}, \gamma^\mu, \epsilon^\mu, u, \upsilon)}{\prod_{\{i_1, i_2, i_3\} \in I} D_{i_1}(k_1) D_{i_2}(k_2) D_{i_3}(k_1, k_2)}$$

[1] S. Pozzorini, N. Schär and M. Zoller, JHEP 05 (2022) 161

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2) Reduction to a set of Master Integrals (or special functions (Dmitry's talk) or tensor integrals):

- *Integrand Reduction (Numerical Unitarity [H. Ita, Phys.Rev.D 94 (2016) 11, 116015], OpenLoops*[1]*)*
- *IBP Reduction + Finite Fields (KIRA, FIRE, Reduze, FiniteFlow, FireFly)*
- *Hybrid approach (AID [P. Mastrolia, T. Peraro and A. Primo, JHEP 08 (2016) 164] + IBP)*

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3) Computation of the Master Integrals:

- *Analytical (Differential Equations (Stefan's talk), SDE approach, Feynman Parametrization)*
- *Numerical (feyntrop (Henrik's talk), Sector Decomposition → pySecDec (Vitalii's talk), FIESTA)*
- *Semi-Numerical (DiffExp, SeaSyde, AMFlow (Xiao's talk), Internal reduction, DiffExp + Feynman trick)*

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DEMOKRITOS

HELAC-2LOOP for amplitude construction: The algorithm

$$n - particle,\ 2 - loop\ Amplitude \longrightarrow (n+2) - particle,\ 1 - loop\ Amplitude$$

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n − particle, 2 − loop Amplitude $\longrightarrow$ $(n+2)$ − particle, 1 − loop Amplitude

1) Definition of the process at hand: number (n) and flavor of the external particles.
2) DO-loop over the type of grand blob-topologies (*Theta* $=1$, *Infinity* $=2$, *Dumbbell* $=3$).
3) DO-loop over the flavor of the $n+1$ and $n+2$ cut-particles. For each n+2 process we define the number of color-states (Color-Flow Representation $\rightarrow n_{q_1}!$) and start a DO-loop over them.
4) Generate all the blob-topologies (for the corresponding type) and start a DO-loop over them.
5) Cut the two-loop ($n-$particle) blob-topology and uniquely correspond it to a one-loop ($n+2-$particle) blob-topology. Thetas are cut in k_3-line while Infinities/Dumbbells in k_2-line.
6) Dress with flavor/color the one-loop blob-topology and cut it $\longrightarrow$ tree-level configuration with n+4 particles! The color configuration is rearranged appropriately after the second cut.
7) Create the currents contributing to the configuration at hand, by applying Dyson-Schwinger recursion to the blobs.
8) Reduce, making contractions with $\delta^{i_{n+3}}_{j_{n+4}}\delta^{i_{n+4}}_{j_{n+3}}\delta^{i_{n+1}}_{j_{n+2}}\delta^{i_{n+2}}_{j_{n+1}}$, the $n+4$ color-state to the corresponding n color-state, and identify the N_C power coming from the contractions.
9) Store the numerator information in the Skeleton and continue with the next configuration.

DEMOKRITOS

Binary Representation and Blobs

- As in HELAC-1LOOP, for the external particles we use a binary representation (2^{i-1}). E.g.:

$$\text{For } n = 4: \quad \{p_1, p_2, p_3, p_4\} \rightarrow \{1, 2, 4, 8\}.$$

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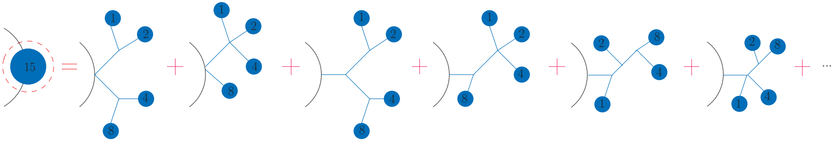

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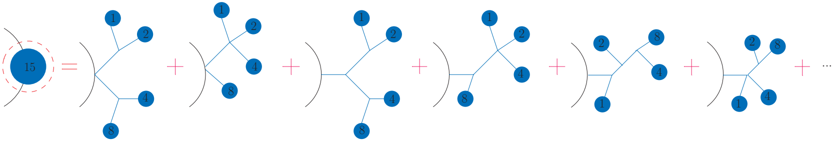

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2) The level of the blob is defined as the number of particles that consists.

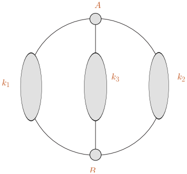
- In the example above where we study the blob 15, its level is 4, and the total number of graphs that describes are 26:

- 3 graphs of the first type,
- 4 graphs of the second type,
- 3 graphs of the third type,
- 4 graphs of the fourth type,
- 12 graphs of the fifth type,
- 3 graphs of the sixth type.

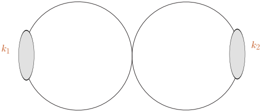
DEMOKRITOS

Two-loop blob-topologies

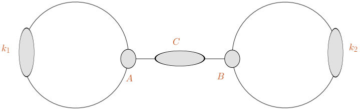
- At two-loop 3 grand blob-topologies exist:

Theta—topologies: 

$\equiv \{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\}$

Infinity—topologies: 

$\equiv \{\{k_1\},\{k_2\}\}$

Dumbbell—topologies: 

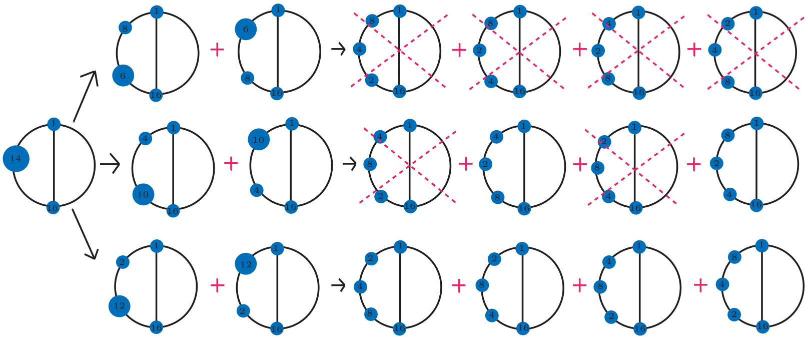
$\equiv \{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\}$

- The sub-lists $(\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\},\{C\})$ represent the incoming blobs to the corresponding loop-lines (k_1, k_2, k_3), the vertex-points (A, B) and the internal line (C).

DEMOKRITOS

Blob-Topology generation

Creation of a `fortran`-based generator ⟶ `GENTOOLS`

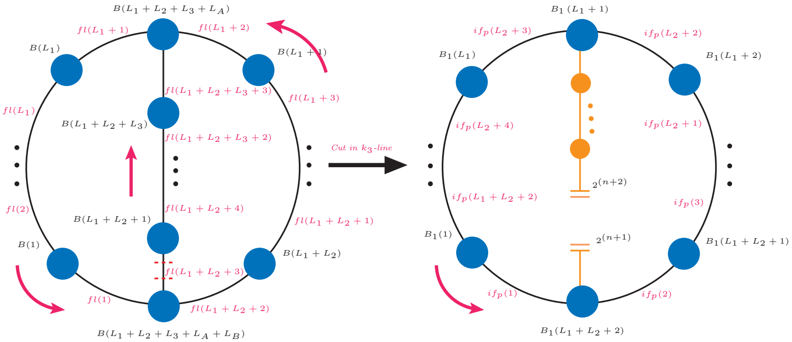


- Generation of all possible blob-topologies: From higher to lower level blobs.
- Order on the Lengths: $L_1 \geq L_2 \geq L_3$ and $L_A \geq L_B$.
- Remove of identical topoes using symmetries: 1) up-down (reversion), 2) loop-line swapping.

DEMOKRITOS

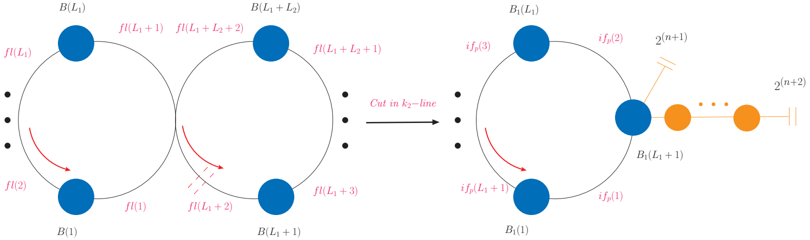
From Two-Loop to One-Loop: The Theta-Topologies

- The red arrows in the following graphs indicate the flow of flavor:



- The orange blobs are the structure information stored for the blobs:
  - $B_1(L_1+1) = 2^{n+2} + B(L_1+L_2+1) + \cdots + B(L_1+L_2+L_3) + B(L_1+L_2+L_3+L_A)$.
  - $B_1(L_1+L_2+2) = 2^{n+1} + B(L_1+L_2+L_3+L_A+L_B)$.
- The rest of the blobs B_1 and the flavors if_p are defined by Bs and fls, respectively, via the relations:
  - For $1 \le i \le L_1$: $B_1(i) = B(i)$, $if_p(1) = fl(1)$ and $if_p(L_1+L_2+3-i) = fl(i+1)$.
  - For $1 \le i \le L_2+1$: $B_1(L_1+1+i) = B(L_1+i)$ (till L_2 no L_2+1) and $if_p(L_2+3-i) = fl(L_1+1+i)$.

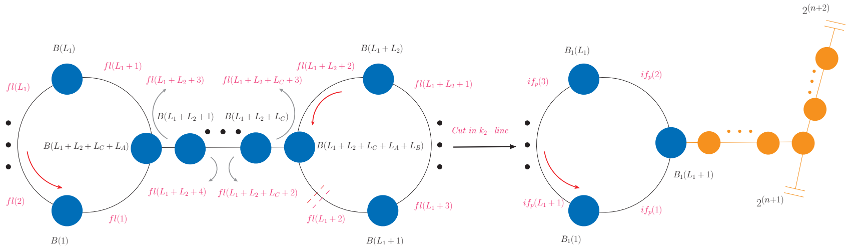
From Two-Loop to One-Loop: The Infinity-Topologies



- The orange blobs are the structure information stored for the blob:
  - $B_1(L_1+1) = 2^{n+1} + 2^{n+2} + B(L_1+1) + \cdots + B(L_1+L_2)$.
- The rest of the blobs B_1 and the flavors if_p are defined by Bs and fls, respectively, via the relations:
  - For $1 \leq i \leq L_1$: $B_1(i) = B(i)$, $if_p(1) = fl(1) = 35$ (gluon) and $if_p(L_1+2-i) = fl(i+1)$.

From Two-Loop to One-Loop: The Dumbbell-Topologies

- The grey arrows indicate to which propagator the pointing flavor corresponds.



- The orange blobs are the structure information stored for the blob:
  - $B_1(L_1{+}1) = 2^{n+1}{+}2^{n+2}{+}B(L_1{+}1){+}\cdots{+}B(L_1{+}L_2{+}L_C){+}B(L_1{+}L_2{+}L_C{+}L_A){+}B(L_1{+}L_2{+}L_C{+}L_A{+}L_B)$.
- The rest of the blobs B_1 and the flavors if_p are defined by Bs and fls, respectively, via the relations:
  - For $1 \leq i \leq L_1$: $B_1(i) = B(i)$, $if_p(1) = fl(1)$ (gluon) and $if_p(L_1 + 2 - i) = fl(i + 1)$.

DEMOKRITOS

Color-Flavor dressing

QCD particles on the loop + Color-Flow representation!

- We dress with flavor and color the loop-particles of the one-loop blob topology using suitable subroutines that apply the following procedures for the dressing:

Flavor Dressing:

- Identification of blob-flavors
- Assign flavor to the first propagator.
- QCD Feynman rules in vertices.

Color Dressing:

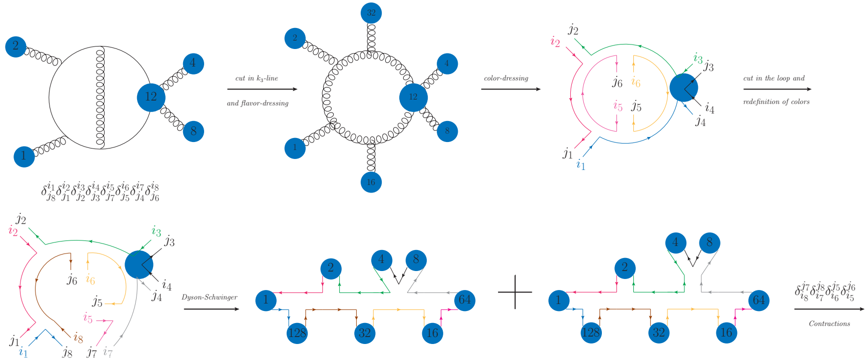
- Assign color and anti-color indices to the first propagator.
- Track the color flow at each vertex.

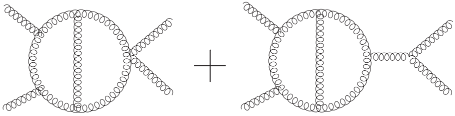
- We also dress with flavor the loop particles of the cut loop-line (the orange one in the graphs of the previous slides).

- After the second cut, we define the flavor and color of the new extra particles and we rearrange the color of the $n + 4$ particle configuration by tracking the flow of color in the one-loop topology.

In this way we obtain a unique configuration with specific {color, anti-color} and flavor for each loop particle!

Construction: gluonic $\{\{1,2\},\{12\},\{\},\{\},\{\}\}$ with $\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}\delta^{i_5}_{j_6}\delta^{i_6}_{j_5}$





with $C_F = N_c^2\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}$

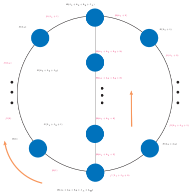
Results: gluonic $\{\{1,2\},\{12\},\{\},\{\},\{\}\}$ with $\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}$

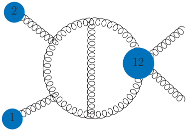
```
INFO ============================================
INFO COLOR            9 out of          24
INFO number of nums           0
INFO ============================================
INFO COLOR           10 out of          24
INFO number of nums         208
```

→ Skeleton stored color-wise

```
 INFO NUM          52   of           208           7
 INFO ====================================
INFO    4  80  35   9   1   1  16  35   5  64  35   7   0   0   0   0   1   2
INFO    4  12  35  10   1   1   4  35   3   8  35   4   0   0   0   0   1   1
INFO    4  92  35  11   1   2  12  35  10  80  35   9   0   0   0   0   1   1
INFO    5  92  35  11   2   2   4  35   3   8  35   4  80  35   9   0   1   5
INFO    4 124  35  12   1   1  32  35   6  92  35  11   0   0   0   0   1   2
INFO    4 126  35  13   1   1   2  35   2 124  35  12   0   0   0   0   1   1
INFO    4 254  35  14   1   1 128  35   8 126  35  13   0   0   0   0   1   2
INFO    6   1  12   1   2  12  35  35  35  35  35  35   0   0   0   0  99   9
```

$$ID = \begin{cases} (2)^{L_1}(3)^{L_2}(5)^{L_3}(7)^{L_A}(11)^{L_B}, & \text{Theta} \\ (2)^{L_1}(3)^{L_2}, & \text{Infinity} \\ (2)^{L_1}(3)^{L_2}(5)^{L_C}(7)^{L_A}(11)^{L_B}, & \text{Dumbbell} \end{cases} \qquad loopnum = \begin{cases} 1, & \text{Theta} \\ 2, & \text{Infinity} \\ 3, & \text{Dumbbell} \end{cases}$$





Numerators and numerics in 4 dimensions

Process	#	Loop-Flavors	Color	Size	Crea. Time	Num/s
$gg \to gg$	2	$\{g, c, \bar{c}\}$	Lead.	8.9 MB	15.017s	4560
$gg \to gg$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	110.6 MB	6m 54.574s	89392
$gg \to q\bar{q}$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	16.1 MB	3m 14.509s	13856
$gg \to ggg$	2	$\{g, c, \bar{c}\}$	Lead.	300.0 MB	21m 42.609s	81480
$gg \to gg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	537.8 kB	2.386s	768
$gg \to ggg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	15.1 MB	8m 53.349s	11496
$gg \to gggg$	1	$\{g, c, \bar{c}\}$	Lead.	394.0 MB	104m 14.95s	19680

[2] $p_{1[2]} = (250, 0, 0, [-]250)$, $p_3 = (250, 49, -176, -171)$, $k_1 = (0.2, 0.3, 0.5, 0.7)$ and $k_3 = (0.9, 0.11, 0.13, 0.15)$

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Comments on the skeletons:

1. n increase, leading color to full color ⟶ complexity increase
2. Timings a bit large ⟶ Skeleton constructed only once per process!
3. Much numerators (some are identical) ⟶ Room for improving efficiency!
4. Comparable (in terms of complexity) 1-loop and 2-loop processes ⟶ 2-loop construction faster!

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Some numerical results for numerators with gluons as external and loop particles ($h = -- \to --$)[2]:

1. $N_{\{\{1,2\},\{12\},\{\},\{\},\{\}\}} = 17052219.315419123 + 64639250.888367772i.$
2. $N_{\{\{1,2\},\{4,8\},\{\},\{\},\{\}\}} = -12231870819598.090 + 5124375444085.5430i.$
3. $N_{\{\{1,2\},\{4\},\{8\},\{\},\{\}\}} = -1268111397619.5310 + 195312105699.88257i.$
4. $N_{\{\{2,1\},\{8\},\{\},\{4\},\{\}\}} = -49731029299.352333 + 15599344.440385548i.$

- Perfect agreement in cross-checks with `FeynArts` + `FeynCalc` + `FORM`!

[2] $p_{1[2]} = (250, 0, 0, [-]250)$, $p_3 = (250, 49, -176, -171)$, $k_1 = (0.2, 0.3, 0.5, 0.7)$ and $k_3 = (0.9, 0.11, 0.13, 0.15)$

DEMOKRITOS

Amplitude Reduction: Work in progress – General Concept

- In general, a 2−loop n−particle amplitude integrand depends on $n_2 = 11$ loop scalar products

$$1)k_i \cdot k_j \rightarrow \#_1 = 3, \quad 2)k_i \cdot p_j \rightarrow \#_2 = \min[4, n-1] \times 2, \quad 3)k_i \cdot \eta_j \rightarrow \#_3 = 8 - \#_2, \quad \text{with} \quad \eta_i \perp p_j .$$

- For a topology I with n_I propagators, n_I scalar products can be expressed as linear combinations of them (*reducible scalar products*) $\rightarrow$ $n_{ir} = 11 - n_I$ *irreducible scalar products* $\rightarrow$ $\{\bar{z}_1, \ldots, \bar{z}_{n_{ir}}\}$.

- The loop momenta can be decomposed into a 4-dimensional ($\bar{k}_i$) and an ε-dimensional part (k_i^*)

$$k_i = \bar{k}_i + k_i^* \qquad \text{with} \qquad k_i \cdot k_j = \bar{k}_i \cdot \bar{k}_j + \mu_{ij}, \qquad \text{and} \qquad \mu_{ij} = k_i^* \cdot k_j^* . \tag{3.1}$$

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- For each numerator configuration we make the following ansatz

$$N_I = \sum_{m=1}^{n_I} \left(\sum_{\mathbf{i_m}} c_{\mathbf{i_m}} \prod_{i \notin \mathbf{i_m}} D_i \right) \qquad \text{with} \qquad \mathbf{i_m} = \{i_1, \cdots, i_{m-1}, i_m\}, \tag{3.2}$$

where the coefficients $c_{\mathbf{i_m}}$ have the following form

$$c_{\mathbf{i_m}} = \sum_{j=1} \tilde{c}^{(j)}_{\mathbf{i_m}}(\vec{s}, d) \left(\bar{z}_1^{(\mathbf{i_m})}\right)^{\alpha_1^{(j)}} \cdots \left(\bar{z}_{n_{ir}}^{(\mathbf{i_m})}\right)^{\alpha_{n_{ir}}^{(j)}} \tag{3.3}$$

with the exponents $\alpha_i^{(j)}$ to be restricted by power counting.

DEMOKRITOS

- In (3.3) the monomials not containing η_j integrate to Feyman integrals $\xrightarrow{IBP}$ master integrals.
- All the monomials in (3.2) containing at least one η_j to an odd power vanish after integration.
- The monomials consisting even powers of η_j can be turned into terms that vanish after integration using traceless completions [H. Ita, Phys. Rev. D 94, no. 11, 116015 (2016)]

1) $(k_i \cdot \eta_j)^2 \longrightarrow (k_i \cdot \eta_j)^2 - \frac{\mu_{ii}}{d-4}$

2) $(k_{i_1} \cdot \eta_j)^2 (k_{i_2} \cdot \eta_j)^2 \longrightarrow (k_{i_1} \cdot \eta_j)^2 (k_{i_2} \cdot \eta_j)^2 - \frac{(k_{i_1} \cdot \eta_j)^2 \mu_{i_2 i_2} + (k_{i_2} \cdot \eta_j)^2 \mu_{i_1 i_1} + 4(k_{i_1} \cdot \eta_j)(k_{i_2} \cdot \eta_j)\mu_{i_1 i_2}}{2(d-4)}$

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Following the one-loop paradigm, a possible path within the HELAC-framework for the amplitude reduction could be :

1. Determine the 4-dimensional part of the coefficients $\tilde{c}^{(j)}_{\mathbf{i_m}}(\vec{s}, d)$ of Eq. (3.3) using values for the loop-momenta obtained from the cut equations $D_{i_1} = \cdots = D_{i_m} = 0$ in an OPP-like approach [G. Ossola, C. G. Papadopoulos, R. Pittau, Nucl.Phys.B 763 (2007)].
2. Determine the ε-dimensional part using two-loop rational terms. The ones of UV origin have been studied in [J. Lang, S. Pozzorini, H. Zhang, M. Zoller, JHEP 05 (2020) 077, JHEP 10 (2020) 016, JHEP 01 (2022) 105].
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DEMOKRITOS

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Main Bottlenecks:

- IBP reduction for two-loop 5-point Feynman integrals could be a time-consuming task.
- Rational terms of IR origin?

DEMOKRITOS

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Conclusion

Results

- Implementation of an algorithm for the construction of two-loop integrand numerators using a hybrid Dyson-Schwinger recursion!

- Validation of the results for numerators including gluons, ghosts and quarks on the loop, with the packages QGRAF, FeynArts, FeynCalc, and FORM!

Next milestones for HELAC-2LOOP

- Development of the part of HELAC-2LOOP for the determination of the 4-dimensional part of the coefficients of the ansatz in Eqs. (3.2) - (3.3).

- Implementation to the current framework of the two-loop rational terms.

- Incorporation of IBP reductions to master integrals of the integrals resulted from the ansatz in Eqs. (3.2) - (3.3) and the traceless completions, from IBP packages like FIRE, KIRA.

- Computation of master integrals and creation of a library for efficient calculations / or use of the pentagon functions library!

Thank you!

The research project was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the 2nd Call for H.F.R.I. Research Projects to support Faculty Members & Researchers (Project Number: 2674).

Color connection representation

- In the color connection representation, the gluons are represented by a pair of color/anti-color indices (i, j) and the quarks (anti-quarks) by a single color $(i, 0)$ (anti-color $(0, j)$) index, with $i_{,j} \in (1, \ldots, N_C)$. All the other particles that do not carry color have $(0, 0)$.

- The amplitude takes the following form

$$\mathcal{M}^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k} = \sum_{\sigma} \delta_{i_{\sigma_1}, j_1} \delta_{i_{\sigma_2}, j_2} \ldots \delta_{i_{\sigma_k}, j_k} A_\sigma$$

with $k = n_g + n_q$ and the sum is running over all the permutations (equal to $k!$). The color-stripped amplitudes, A_σ, are calculated using properly defined Feynman rules [A. Cafarella, C. G. Papadopoulos and M. Worek, Comput. Phys. Commun. **180** (2009), 1941-1955].

- The total color factor is a product of δ's, and thus the color summed squared amplitude takes the form

$$\sum_{\{i\},\{j\}} \left| \mathcal{M}^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k} \right|^2 = \sum_{\sigma, \sigma'} A^*_{\sigma'} \mathcal{C}_{\sigma', \sigma} A_\sigma$$

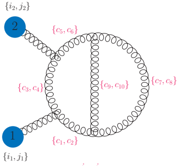
where the color matrix $C_{\sigma', \sigma}$ is given by

$$C_{\sigma', \sigma} = \sum_{\{i\},\{j\}} \delta_{i_{\sigma'_1}, j_1} \delta_{i_{\sigma'_2}, j_2} \ldots \delta_{i_{\sigma'_k}, j_k} \delta_{i_{\sigma_1}, j_1} \delta_{i_{\sigma_2}, j_2} \ldots \delta_{i_{\sigma_k}, j_k} = N_C^{m(\sigma', \sigma)}$$

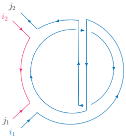
with $m(\sigma', \sigma)$ counting the number of common cycles of the 2 permutations.

DEMOKRITOS

Two-loop color-flow dressing → identical configurations for HELAC

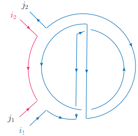


$\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$



$\{c_1, c_2\} = \{i_1, i_1\}$
$\{c_3, c_4\} = \{i_2, i_1\}$
$\{c_5, c_6\} = \{i_1, i_1\}$
$\{c_7, c_8\} = \{i_1, i_1\}$
$\{c_9, c_{10}\} = \{i_1, i_1\}$

$$C_F = -\delta^{i_1}_{c_1}\delta^{c_1}_{c_7}\delta^{c_7}_{c_{10}}\delta^{c_{10}}_{c_2}\delta^{c_2}_{c_4}\delta^{c_4}_{c_6}\delta^{c_6}_{c_8}\delta^{c_8}_{c_9}\delta^{c_9}_{c_5}\delta^{c_5}_{j_2}\delta^{i_2}_{c_3}\delta^{c_3}_{j_1}$$

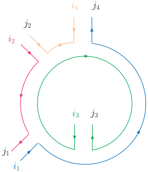


$\{c_1, c_2\} = \{i_1, i_1\}$
$\{c_3, c_4\} = \{i_2, i_1\}$
$\{c_5, c_6\} = \{i_1, i_1\}$
$\{c_7, c_8\} = \{i_1, i_1\}$
$\{c_9, c_{10}\} = \{i_1, i_1\}$

$$C_F = -\delta^{i_1}_{c_1}\delta^{c_1}_{c_9}\delta^{c_9}_{c_8}\delta^{c_8}_{c_2}\delta^{c_2}_{c_4}\delta^{c_4}_{c_6}\delta^{c_6}_{c_{10}}\delta^{c_{10}}_{c_7}\delta^{c_7}_{c_5}\delta^{c_5}_{j_2}\delta^{i_2}_{c_3}\delta^{c_3}_{j_1}$$

This is not any more the case after cutting in k_3-line:

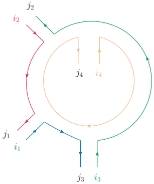
1)



$\{c_1, c_2\} = \{i_1, i_3\}$
$\{c_3, c_4\} = \{i_2, i_3\}$
$\{c_5, c_6\} = \{i_4, i_3\}$
$\{c_7, c_8\} = \{i_1, i_3\}$

$\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}\delta^{i_4}_{j_2}$

2)



$\{c_1, c_2\} = \{i_1, i_4\}$
$\{c_3, c_4\} = \{i_2, i_4\}$
$\{c_5, c_6\} = \{i_3, i_4\}$
$\{c_7, c_8\} = \{i_3, i_4\}$

$\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_4}$

DEMOKRITOS

Blob-Topology Symmetries

- Theta-Topology symmetries[3]:

$$\begin{aligned}\{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\} &= \{R[\{k_1\}],R[\{k_2\}],R[\{k_3\}],\{B\},\{A\}\} = \{\{k_1\},\{k_3\},\{k_2\},\{A\},\{B\}\}\\ &= \{R[\{k_1\}],R[\{k_3\}],R[\{k_2\}],\{B\},\{A\}\} = \{\{k_2\},\{k_1\},\{k_3\},\{A\},\{B\}\}\\ &= \{R[\{k_2\}],R[\{k_1\}],R[\{k_3\}],\{B\},\{A\}\} = \{\{k_2\},\{k_3\},\{k_1\},\{A\},\{B\}\}\\ &= \{R[\{k_2\}],R[\{k_3\}],R[\{k_1\}],\{B\},\{A\}\} = \{\{k_3\},\{k_1\},\{k_2\},\{A\},\{B\}\}\\ &= \{R[\{k_3\}],R[\{k_1\}],R[\{k_2\}],\{B\},\{A\}\} = \{\{k_3\},\{k_2\},\{k_1\},\{A\},\{B\}\}\\ &= \{R[\{k_3\}],R[\{k_2\}],R[\{k_1\}],\{B\},\{A\}\}\end{aligned}$$

- Infinity-Topology symmetries:

$$\begin{aligned}\{\{k_1\},\{k_2\}\} &= \{\{k_2\},\{k_1\}\} = \{R[\{k_1\}],\{k_2\}\} = \{\{k_1\},R[\{k_2\}]\} = \{R[\{k_1\}],R[\{k_2\}]\}\\ &= \{R[\{k_2\}],\{k_1\}\} = \{\{k_2\},R[\{k_1\}]\} = \{R[\{k_2\}],R[\{k_1\}]\}\end{aligned}$$

- Dumbbell-Topology symmetries:

$$\begin{aligned}\{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\} &= \{R[\{k_1\}],\{k_2\},\{C\},\{A\},\{B\}\} = \{\{k_1\},R[\{k_2\}],\{C\},\{A\},\{B\}\}\\ &= \{R[\{k_1\}],R[\{k_2\}],\{C\},\{A\},\{B\}\} = \{\{k_2\},\{k_1\},R[\{C\}],\{B\},\{A\}\}\\ &= \{R[\{k_2\}],\{k_1\},R[\{C\}],\{B\},\{A\}\} = \{\{k_2\},R[\{k_1\}],R[\{C\}],\{B\},\{A\}\}\\ &= \{R[\{k_2\}],R[\{k_1\}],R[\{C\}],\{B\},\{A\}\}\end{aligned}$$

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[3] We use the notation $R[\{k_i\}] := Reverse[\{k_i\}]$.