QCD higher-order corrections: current status and prospects

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HOCTools-II

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Outline

1. Introduction: what calculations we need
2. LO: from Feynman diagrams to recursive equations
3. The NLO revolution: from Feynman Integrals to integrands
4. Towards higher precision: NNLO $2 \to 3$, N^{3}LO $2 \to 2$
5. Summary - Discussion

INTRODUCTION

Higgs

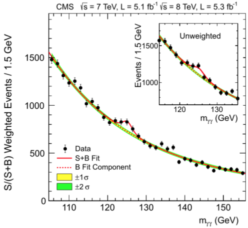


Fig. 3. The diphoton invariant mass distribution with each event weighted by the $S/(S+B)$ value of its category. The lines represent the fitted background and signal, and the coloured bands represent the ±1 and ±2 standard deviation uncertainties in the background estimate. The inset shows the central part of the unweighted invariant mass distribution. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this Letter.)

Gravitational wave

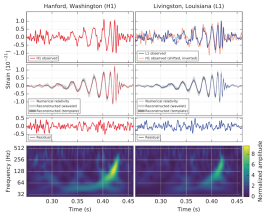


FIG. 1. The gravitational-wave event GW150914 observed by the LIGO Hanford (H1, left column panels) and Livingston (L1, right column panels) detectors. Times are shown relative to September 14, 2015 at 09:50:45 UTC. For visualization, all time series are filtered with a 35–350 Hz bandpass filter to suppress large fluctuations outside the detectors' most sensitive frequency band, and band-reject filters to remove the strong instrumental spectral lines seen in the Fig. 3 spectra. *Top row, left:* H1 strain. *Top row, right:* L1 strain. GW150914 arrived first at L1 and $6.9^{+0.5}_{-0.4}$ ms later at H1; for a visual comparison, the H1 data are also shown, shifted in time by this amount and inverted (to account for the detectors' relative orientations). *Second row:* Gravitational-wave strain projected onto each detector in the 35–350 Hz band. Solid lines show a numerical relativity waveform for a system with parameters consistent with those recovered from GW150914 [37,38] confirmed to 99.9% by an independent calculation based on [15]. Shaded areas show 90% credible regions for two independent waveform reconstructions. One (dark gray) models the signal using binary black hole template waveforms [39]. The other (light gray) does not use an astrophysical model, but instead calculates the strain signal as a linear combination of sine-Gaussian wavelets [40,41]. These reconstructions have a 94% overlap, as shown in [39]. *Third row:* Residuals after subtracting the filtered numerical relativity waveform from the filtered detector time series. *Bottom row:* A time-frequency representation [42] of the strain data, showing the signal frequency increasing over time.

BH Horizon

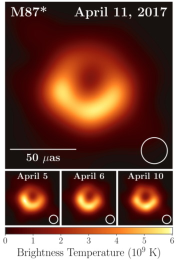
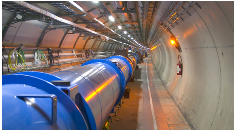
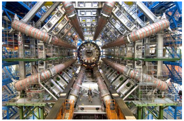


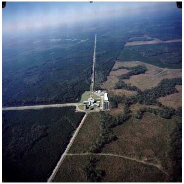
Figure 3. Top: EHT image of M87* from observations on 2017 April 11 as a representative example of the images collected in the 2017 campaign. The image is the average of three different imaging methods after convolving each with a circular Gaussian kernel to give matched resolutions. The largest of the three kernels (20 μas FWHM) is shown in the lower right. The image is shown in units of brightness temperature, $T_b = S\lambda^2/2k_B\Omega$, where S is the flux density, λ is the observing wavelength, k_B is the Boltzmann constant, and Ω is the solid angle of the resolution element. Bottom: similar images taken over different days showing the stability of the basic image structure and the equivalence among different days. North is up and east is to the left.

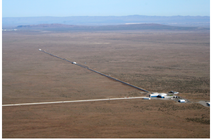
INTRODUCTION

LHC	LIGO	EHT









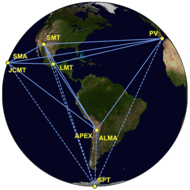
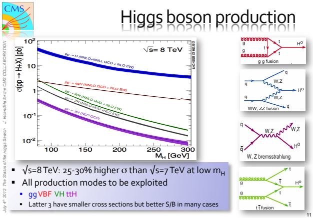


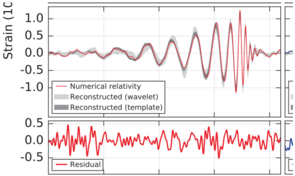
Figure 1. Eight stations of the EHT 2017 campaign over six geographic locations as viewed from the equatorial plane. Solid baselines represent mutual visibility on M87* (+12° declination). The dashed baselines were used for the calibration source 3C279 (see Papers III and IV).

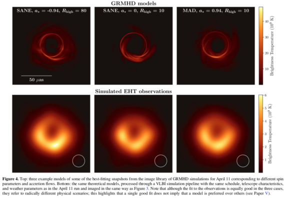
LHC

LIGO

EHT

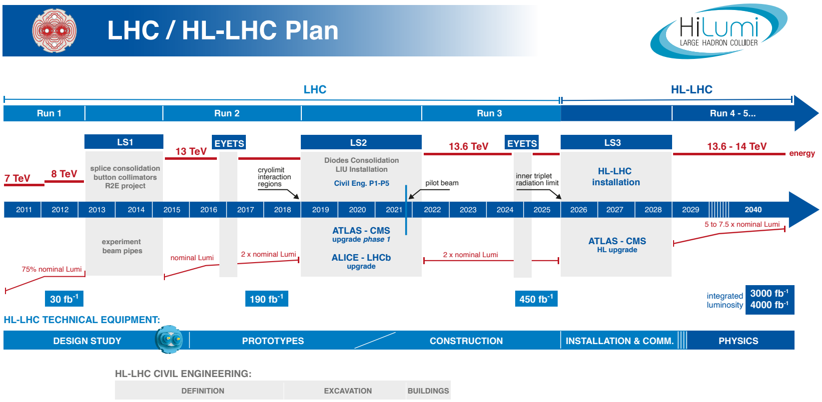




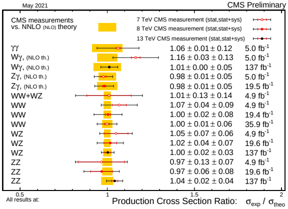


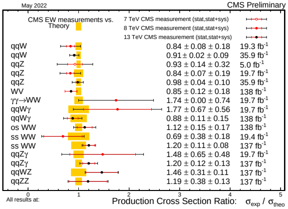
Faint signals; Patience; Theory

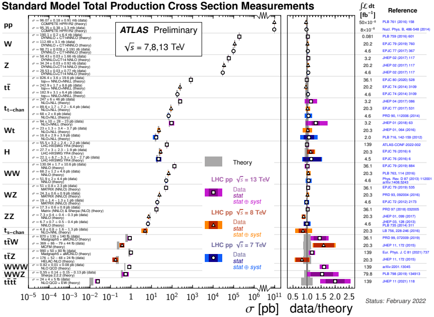
LHC SCHEDULE

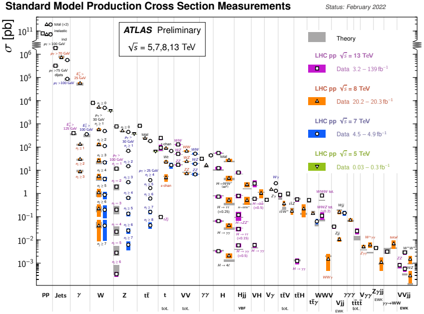


LHC PRECISION



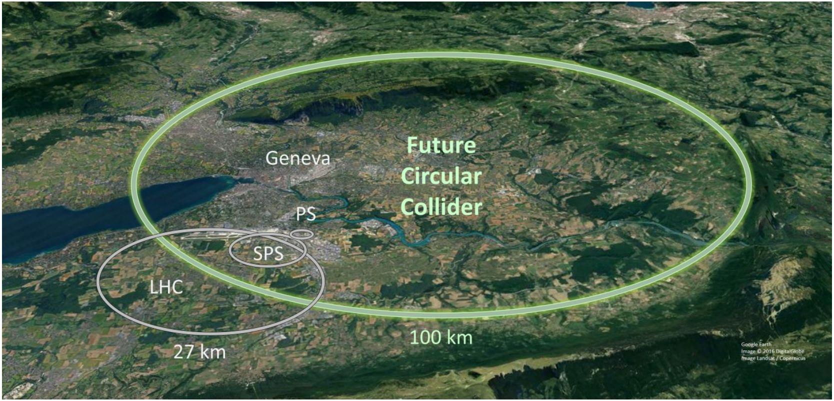






Improved theoretical predictions are indispensable

FCC



circular accelerators with decelerating pace of expansion!

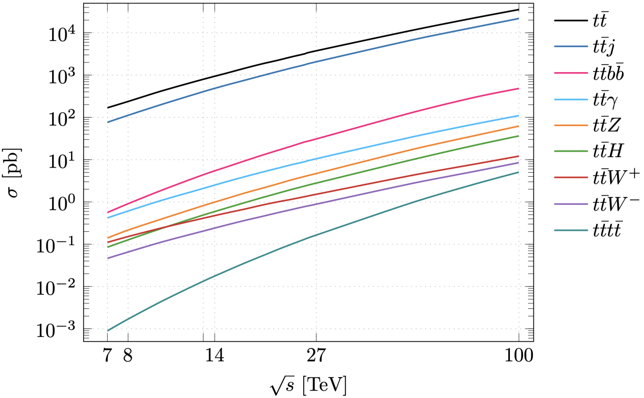
FCC-hh rates - Higgs

→ A. Abada *et al.* [FCC], Eur. Phys. J. ST **228** (2019) no.4, 755-1107

Table 1.1: Higgs production event rates for selected processes at 100 TeV (N_{100}) and statistical increase with respect to the statistics of the HL-LHC ($N_{100} = \sigma_{100\ \mathrm{TeV}} \times 30\ \mathrm{ab}^{-1}$, $N_{14} = \sigma_{14\ \mathrm{TeV}} \times 3\ \mathrm{ab}^{-1}$).

	$gg \to H$	VBF	WH	ZH	$t\bar{t}H$	HH
N_{100}	24×10^9	2.1×10^9	4.6×10^8	3.3×10^8	9.6×10^8	3.6×10^7
N_{100}/N_{14}	180	170	100	110	530	390

FCC-hh rates - top

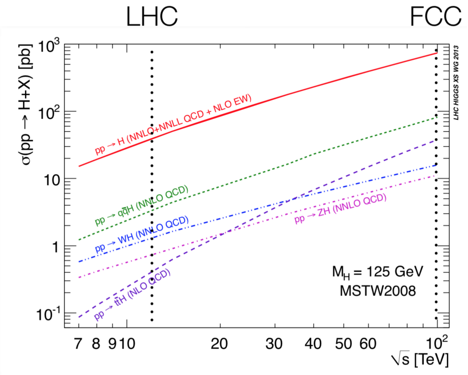


Courtesy of Manfred Kraus/Malgorzata Worek

FCC-HH RATES

→ Talk by Heather M. Gray

Process	σ(100 TeV)/σ(14 TeV)
Total pp	1.25
W,Z	~7
WW,ZZ	~10
tt	~30
H	**~15**
ttH	**~60**
HH	**~40**
stop (m=1 TeV)	~1000



Fixed-order precision frontier

→ A. Huss, J. Huston, S. Jones and M. Pellen, [arXiv:2207.02122 [hep-ph]].

→ M. Begel, *et al.* [arXiv:2209.14872 [hep-ph]].

process	known	desired
$pp \to H$	$\mathrm{N^3LO_{HTL}}$, $\mathrm{NNLO^{(t)}_{QCD}}$, $\mathrm{N^{(1,1)}LO^{(HTL)}_{QCD\otimes EW}}$	$\mathrm{N^4LO_{HTL}}$ (incl.), $\mathrm{NNLO^{(b,c)}_{QCD}}$
$pp \to H + j$	$\mathrm{NNLO_{HTL}}$, $\mathrm{NLO_{QCD}}$, $\mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$	$\mathrm{NNLO_{HTL}} \otimes \mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to H + 2j$	$\mathrm{NLO_{HTL}} \otimes \mathrm{LO_{QCD}}$ $\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ (incl.), $\mathrm{NNLO^{(VBF^*)}_{QCD}}$, $\mathrm{NLO^{(VBF)}_{EW}}$	$\mathrm{NNLO_{HTL}} \otimes \mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$, $\mathrm{NNLO^{(VBF)}_{QCD}}$
$pp \to H + 3j$	$\mathrm{NLO_{HTL}}$, $\mathrm{NLO^{(VBF)}_{QCD}}$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to VH$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$, $\mathrm{NLO^{(t,b)}_{gg\to HZ}}$	
$pp \to VH + j$	$\mathrm{NNLO_{QCD}}$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to HH$	$\mathrm{N^3LO_{HTL}} \otimes \mathrm{NLO_{QCD}}$	$\mathrm{NLO_{EW}}$
$pp \to HHH$	$\mathrm{NNLO_{HTL}}$	
$pp \to H + t\bar{t}$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$, $\mathrm{NNLO_{QCD}}$ (off-diag.)	$\mathrm{NNLO_{QCD}}$
$pp \to H + t/\bar{t}$	$\mathrm{NLO_{QCD}}$	$\mathrm{NNLO_{QCD}}$, $\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$

Fixed-order precision frontier

$pp \to V$	$\mathrm{N^3LO_{QCD}}$, $\mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$, $\mathrm{NLO_{EW}}$	$\mathrm{N^3LO_{QCD}} + \mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$, $\mathrm{N^2LO_{EW}}$
$pp \to VV'$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$, $+\mathrm{NLO_{QCD}}$ (gg)	$\mathrm{NLO_{QCD}}$ (gg,massive loops)
$pp \to V + j$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$	hadronic decays
$pp \to V + 2j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$, $\mathrm{NLO_{EW}}$	$\mathrm{NNLO_{QCD}}$
$pp \to V + b\bar{b}$	$\mathrm{NLO_{QCD}}$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to VV' + 1j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$	$\mathrm{NNLO_{QCD}}$
$pp \to VV' + 2j$	$\mathrm{NLO_{QCD}}$ (QCD), $\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (EW)	Full $\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to W^+W^+ + 2j$	Full $\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$	
$pp \to W^+W^- + 2j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (EW component)	
$pp \to W^+Z + 2j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (EW component)	
$pp \to ZZ + 2j$	Full $\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$	
$pp \to VV'V''$	$\mathrm{NLO_{QCD}}$, $\mathrm{NLO_{EW}}$ (w/o decays)	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to W^\pm W^+ W^-$	$\mathrm{NLO_{QCD}}$ + $\mathrm{NLO_{EW}}$	
$pp \to \gamma\gamma$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$	$\mathrm{N^3LO_{QCD}}$
$pp \to \gamma + j$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$	$\mathrm{N^3LO_{QCD}}$
$pp \to \gamma\gamma + j$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$, $+\mathrm{NLO_{QCD}}$ (gg channel)	
$pp \to \gamma\gamma\gamma$	$\mathrm{NNLO_{QCD}}$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$

Fixed-order precision frontier

$pp \to 2\,\text{jets}$	NNLO_{QCD}, $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$	$\text{N}^3\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$
$pp \to 3\,\text{jets}$	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$	
$pp \to t\bar{t}$	NNLO_{QCD} (w/ decays)+ NLO_{EW} (w/o decays) $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays, off-shell) NNLO_{QCD}	$\text{N}^3\text{LO}_{\text{QCD}}$
$pp \to t\bar{t} + j$	NLO_{QCD} (w/ decays, off-shell) NLO_{EW} (w/o decays)	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)
$pp \to t\bar{t} + 2j$	NLO_{QCD} (w/o decays)	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)
$pp \to t\bar{t} + Z$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/o decays) NLO_{QCD} (w/ decays, off-shell)	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)
$pp \to t\bar{t} + W$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays, off-shell)	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)
$pp \to t/\bar{t}$	NNLO_{QCD}*(w/ decays) NLO_{EW} (w/o decays)	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)
$pp \to tZj$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/o decays)

FCC-ee rates

→ A. Blondel, *et al.* [arXiv:1905.05078 [hep-ph]].

TABLE: Run plan for FCC-ee in its baseline configuration with two experiments. The WW event numbers are given for the entirety of the FCC-ee running at and above the WW threshold.

Phase	Run duration (years)	Centre-of-mass energies (GeV)	Integrated luminosity (ab^{-1})	Event statistics
FCC-ee-Z	4	88–95	150	3×10^{12} visible Z decays
FCC-ee-W	2	158–162	12	10^8 WW events
FCC-ee-H	3	240	5	10^6 ZH events
FCC-ee-tt	5	345–365	1.5	10^6 $t\bar{t}$ events

FCC-ee rates

→ D. d'Enterria, [arXiv:1602.05043 [hep-ex]].

$\sqrt{s}$ (GeV):	90 (Z)	125 (eeH)	160 (WW)	240 (HZ)	350 ($t\bar{t}$)	350 (WW→H)
$\mathscr{L}$/IP (cm^{-2} s^{-1})	$2.2{\cdot}10^{36}$	$1.1{\cdot}10^{36}$	$3.8{\cdot}10^{35}$	$8.7{\cdot}10^{34}$	$2.1{\cdot}10^{34}$	$2.1{\cdot}10^{34}$
$\mathscr{L}_{\rm int}$ (ab^{-1}/yr/IP)	22	11	3.8	0.87	0.21	0.21
Events/year (4 IPs)	$3.7{\cdot}10^{12}$	$1.2{\cdot}10^{4}$	$6.1{\cdot}10^{7}$	$7.0{\cdot}10^{5}$	$4.2{\cdot}10^{5}$	$2.5{\cdot}10^{4}$
Years needed (4 IPs)	2.5	1.5	1	3	0.5	3

Table 1: Target luminosities, events/year, and years needed to complete the W, Z, H and top-quark programs at FCC-ee. [Note that $\mathscr{L} = 10^{35}$ cm^{-2} s^{-1} corresponds to $\mathscr{L}_{\rm int} = 1$ ab^{-1}/yr for 1 yr = 10^7 s].

Observable	Measurement	Current precision	FCC-ee stat.	Possible syst.	Challenge
$m_{\rm Z}$ (MeV)	Z lineshape	91187.5 ± 2.1	0.005	< 0.1	QED corr.
$\Gamma_{\rm Z}$ (MeV)	Z lineshape	2495.2 ± 2.3	0.008	< 0.1	QED corr.
R_ℓ	Z peak	20.767 ± 0.025	0.0001	< 0.001	QED corr.
$R_{\rm b}$	Z peak	0.21629 ± 0.00066	0.000003	< 0.00006	$g \to {\rm b\bar{b}}$
N_ν	Z peak	2.984 ± 0.008	0.00004	0.004	Lumi meas.
N_ν	$\rm e^+e^- \to \gamma Z(inv.)$	2.92 ± 0.05	0.0008	< 0.001	–
$A_{\rm FB}^{\mu\mu}$	Z peak	0.0171 ± 0.0010	0.000004	< 0.00001	$E_{\rm beam}$ meas.
$\alpha_s(m_{\rm Z})$	$R_\ell, \sigma_{\rm had}, \Gamma_{\rm Z}$	0.1190 ± 0.0025	0.000001	0.00015	New physics
$1/\alpha_{\rm QED}(m_{\rm Z})$	$A_{\rm FB}^{\mu\mu}$ around Z peak	128.952 ± 0.014	0.004	0.002	EW corr.
$m_{\rm W}$ (MeV)	WW threshold scan	80385 ± 15	0.3	< 1	QED corr.
$\alpha_s(m_{\rm W})$	$\Gamma_{\rm W}, B_{\rm had}^{\rm W}$	$B_{\rm had}^{\rm W} = 67.41 \pm 0.27$	0.00018	0.00015	CKM matrix
$m_{\rm t}$ (MeV)	$t\bar{t}$ threshold scan	173200 ± 900	10	10	QCD
$\Gamma_{\rm t}$ (MeV)	$t\bar{t}$ threshold scan	1410^{+290}_{-150}	12	?	$\alpha_s(m_{\rm Z})$
$y_{\rm t}$	$t\bar{t}$ threshold scan	$\mu = 2.5 \pm 1.05$	13%	?	$\alpha_s(m_{\rm Z})$
$F_{\rm 1V,2V,1A}^{\gamma t, Z t}$	$d\sigma^{t\bar{t}}/dx\, d\cos(\theta)$	4%–20% (LHC-14 TeV)	(0.1–2.2)%	(0.01–100)%	–

FCC-EE THEORY

→Marek Schönherr

Process	Theory	Monte-Carlo
Z-pole	NNLO EW needed throughout (N3LO in some places) including ISR, FSR resummation and initial-final interference (IFI)	highest precision Monte-Carlo event generators to account for finite fiducial region, bremsstrahlung effects, hadronisation corrections, etc.
WW-threshold	needs precision calculation (NNLO QCD, QCD-EW, EW) and QED threshold resummation	including implementation in Monte-Carlo event generators to account for finite ?fiducial region, colour reconnection, hadronisation, etc.
ZH-threshold	direct access to all Higgs decay channels incl. $h \to gg$ and $h \to$ inv.	Monte-Carlo event generators with highest precision for both production mechanisms and Higgs decays necessary
$t\bar{t}$-threshold	needs precision calculation (NNLO QCD, QCD-EW) and QED+QCD threshold resummation	implemented in Monte-Carlo event generators to account for finite fiducial region, top decay kinematics, colour reconnection, hadronisation, etc.

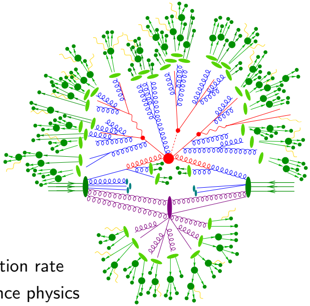
Besides QCD, complicated EW and demanding QED corrections!

Fixed-order calculations

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
  - Gluon emission probability
- Measure
  - Long distance interactions
  - Particle decay rates



Divide et Impera

- Quantity of interest: Total interaction rate
- Convolution of short & long distance physics

$$\sigma_{p_1p_2\to X} = \sum_{i,j\in\{q,g\}} \int \mathrm{d}x_1\mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij\to X}(x_1x_2,\mu_F^2)}_{\text{short distance physics}}$$

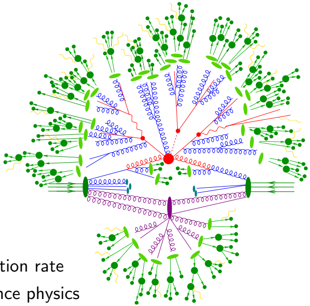
QCD as a perturbative quantum field theory

Factorization

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
  - Gluon emission probability
- Measure
  - Long distance interactions
  - Particle decay rates



Divide et Impera

- Quantity of interest: Total interaction rate
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$$\sigma_{p_1 p_2 \to X} = \sum_{i,j \in \{q,g\}} \int \mathrm{d}x_1 \mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij \to X}(x_1 x_2, \mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory

Lattice QCD results: → C. Alexandrou, *et al.* Phys. Rev. Lett. **121**, 112001 (2018) [arXiv:1803.02685 [hep-lat]].

Leading Order

How to avoid Feynman diagrams

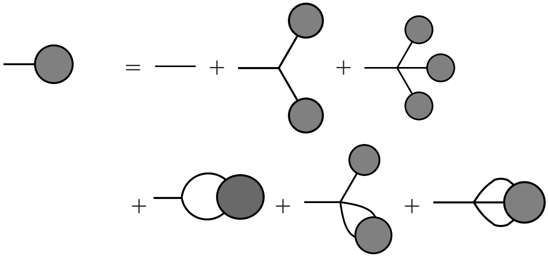
$\rightarrow$ a highly subjective point of view

`MadGraph` → T. Stelzer and W. F. Long, Comput. Phys. Commun. **81**, 357 (1994)

- 1999 `HELAC`: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

From Feynman Diagrams to recursive equations: taming the *n*!

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles



Unfortunately not so much on the second line !

LO - Dyson-Schwinger Recursive Equations

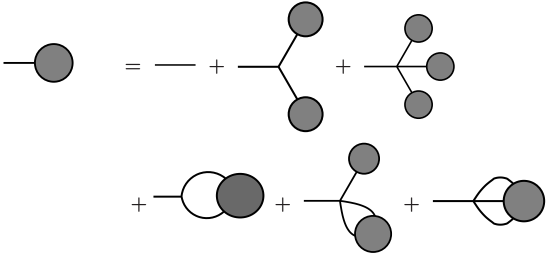
From Feynman Diagrams to recursive equations: taming the $n!$

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

→A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

→F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

→ F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

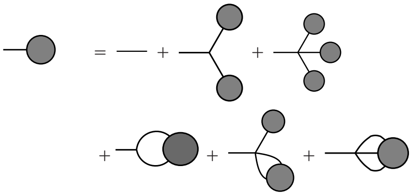
From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

to Dyson-Schwinger recursion! `Helac-Phegas`



$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

NLO

Don't make integrals, make integrands !

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \quad \leftarrow LO \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \quad \leftarrow Virtual \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \quad \leftarrow Real
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned} \sigma_{NLO} &= \int_m d\Phi_m^{D=4}\left(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV}))\right) J_m(\Phi) \\ &+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence μ_R

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1\,,p_2 \rightarrow p_3,\ldots,p_{m+2}$$

$$\begin{aligned}\sigma_{NLO} &= \int_m d\Phi_m J_m(\Phi) \\ &+ \int_m d\Phi_m 2Re(M_m^{(0)*}M_m^{(1)}(\epsilon_{UV},\epsilon_{IR}))J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1}|M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)\end{aligned}$$

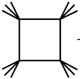
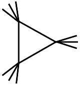
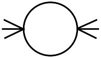
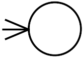
QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved basis of scalar integrals: known already before NLO-R; remember this is not the case for higher orders

→G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

→ Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751

→G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

→Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217.

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

The old "master" formula

$$\mathcal{A} \to \int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}}$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}}$$

$$+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}}$$

$$+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}}$$

$$+ \text{rational terms}$$

OPP "MASTER" FORMULA - I

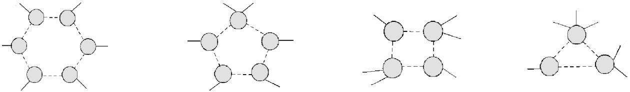
General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

The one-loop calculation in a nutshell

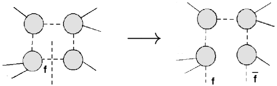
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), ...$ with the values of the loop momentum q provided by CutTools

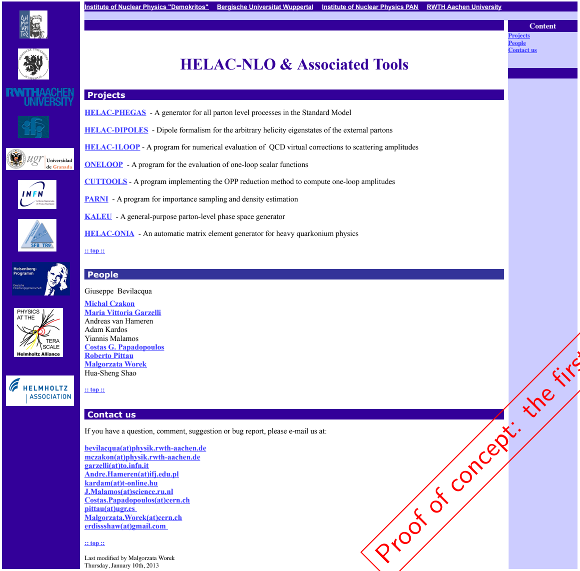
- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

$\to$ BlackHat, MadGraph, RECOLA, OpenLoops

The one-loop calculation in a nutshell


Institute of Nuclear Physics "Demokritos" Bergische Universitat Wuppertal Institute of Nuclear Physics PAN RWTH Aachen University

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HELAC-NLO & Associated Tools

Projects

HELAC-PHEGAS - A generator for all parton level processes in the Standard Model

HELAC-DIPOLES - Dipole formalism for the arbitrary helicity eigenstates of the external partons

HELAC-1LOOP - A program for numerical evaluation of QCD virtual corrections to scattering amplitudes

ONELOOP - A program for the evaluation of one-loop scalar functions

CUTTOOLS - A program implementing the OPP reduction method to compute one-loop amplitudes

PARNI - A program for importance sampling and density estimation

KALEU - A general-purpose parton-level phase space generator

HELAC-ONIA - An automatic matrix element generator for heavy quarkonium physics

:: top ::

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Yiannis Malamos
Costas G. Papadopoulos
Roberto Pittau
Malgorzata Worek
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:: top ::

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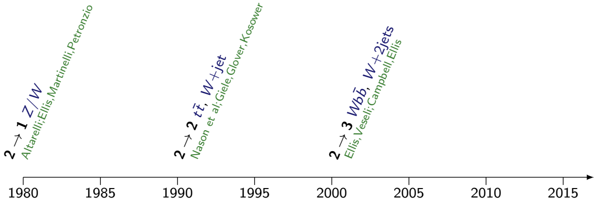
:: top ::

Last modified by Malgorzata Worek
Thursday, January 10th, 2013

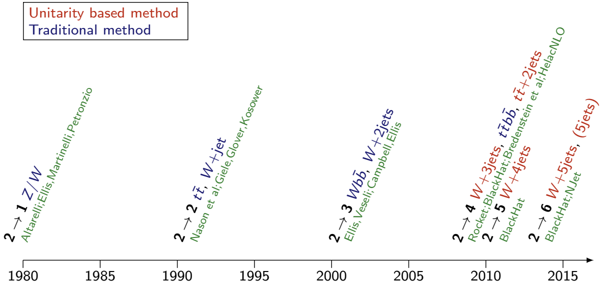
Proof of concept: the first NLO public code

G. P. Salam, PoS ICHEP **2010**, 556 (2010) [arXiv:1103.1318 [hep-ph]]

The NLO revolution



The NLO revolution



BlackHat → Berger,Bern,Dixon,Febres Cordero,Forde,Ita,Kosower,Maître
HelacNLO → Bevilacqua,Czakon,Papadopoulos,Pittau,Worek
NJet → Badger,Biedermann,Uwer,Yundin
Rocket → Ellis,Melnikov,Zanderighi

MadGraph: →J. Alwall *et al.*, JHEP **1407** (2014) 079 [arXiv:1405.0301 [hep-ph]].

OpenLoops: → F. Cascioli, P. Maierhofer and S. Pozzorini, Phys. Rev. Lett. **108**, 111601 (2012) [arXiv:1111.5206 [hep-ph]].

The NLO wishlist

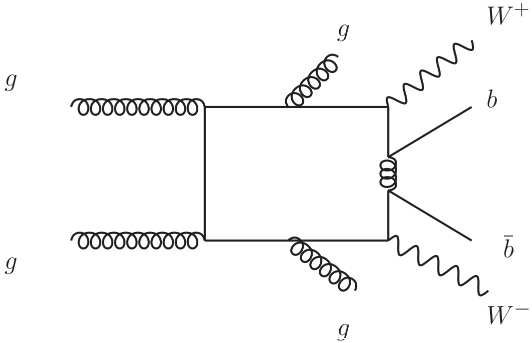
Process ($V \in \{Z, W, \gamma\}$)	Status
1. $pp \to VV$ jet	WW jet completed by Dittmaier/Kallweit/Uwer; Campbell/Ellis/Zanderighi ZZ jet completed by Binoth/Gleisberg/Karg/Kauer/Sanguinetti WZ jet, $W\gamma$ jet completed by Campanario et al.
2. $pp \to$ Higgs+2 jets	NLO QCD to the gg channel completed by Campbell/Ellis/Zanderighi NLO QCD+EW to the VBF channel completed by Ciccolini/Denner/Dittmaier Interference QCD-EW in VBF channel
3. $pp \to V\,V\,V$	ZZZ completed by Lazopoulos/Melnikov/Petriello and WWZ by Hankele/Zeppenfeld see also Binoth/Ossola/Papadopoulos/Pittau VBFNLOmeanwhile also contains WWW, ZZW, ZZZ, $WW\gamma$, $ZZ\gamma$, $WZ\gamma$, $W\gamma\gamma$, $Z\gamma\gamma$, $\gamma\gamma\gamma$, $W\gamma\gamma j$
4. $pp \to t\bar{t}\,b\bar{b}$	relevant for $t\bar{t}H$, computed by Bredenstein/Denner/Dittmaier/Pozzorini and Bevilacqua/Czakon/Papadopoulos/Pittau/Worek
5. $pp \to V$+3 jets	W+3 jets calculated by the Blackhat/Sherpa and Rocket collaborations Z+3jets by Blackhat/Sherpa
6. $pp \to t\bar{t}$+2jets	relevant for $t\bar{t}H$, computed by Bevilacqua/Czakon/Papadopoulos/Worek
7. $pp \to VV\,b\bar{b}$,	Pozzorini et al.Bevilacqua et al.
8. $pp \to VV$+2jets	W^+W^++2jets, W^+W^-+2jets, relevant for VBF $H \to VV$ VBF contributions by (Bozzi/)Jäger/Oleari/Zeppenfeld Binoth et al.
9. $pp \to b\bar{b}b\bar{b}$	
10. $pp \to V+4$ jets	top pair production, various new physics signatures Blackhat/Sherpa: W+4jets, Z+4jets see also HEJfor $W + n$jets
11. $pp \to Wb\bar{b}j$	top, new physics signatures, Reina/Schutzmeier
12. $pp \to t\bar{t}t\bar{t}$	various new physics signatures, Bevilacqua/Worek
$pp \to W\gamma\gamma$ jet	Campanario/Englert/Rauch/Zeppenfeld
$pp \to 4/5$ jets	Blackhat+Sherpa/NJets

Closed

- NLO calculations requested by LHC experimenters
- List constructed in 2005
- Calculations completed 2012

NLO Revolution

→ G. Bevilacqua, M. Lupattelli, D. Stremmer and M. Worek, [arXiv:2212.04722 [hep-ph]].



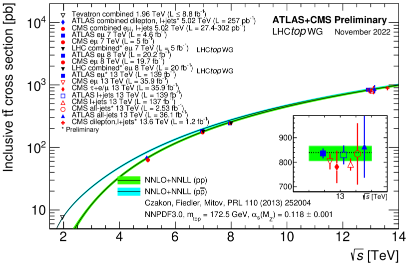
NLO 2 → 6 (2 → 8 including leptonic $W^{\pm}$ decays)

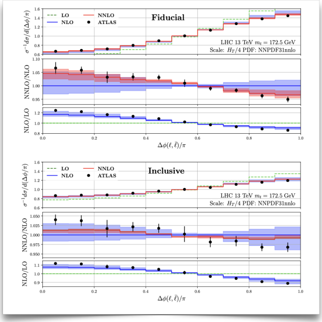
Towards higher precision:

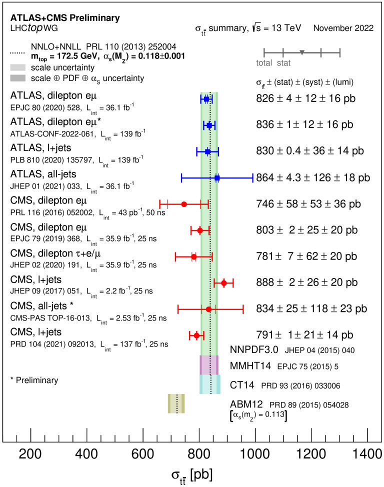
NNLO and beyond

I have a dream ...

The two-loop frontier: 2$\rightarrow$2 @ NNLO


Inclusive tt̄ cross section [pb]
Tevatron combined 1.96 TeV (L ≤ 8.8 fb^{-1})
ATLAS combined dilepton, l+jets* 5.02 TeV (L = 257 pb^{-1})
CMS combined eμ, l+jets 5.02 TeV (L = 27.4-302 pb^{-1})
ATLAS eμ 7 TeV (L = 4.6 fb^{-1})
CMS eμ 7 TeV (L = 5 fb^{-1})
LHC combined* eμ 7 TeV (L = 5 fb^{-1})
ATLAS eμ 8 TeV (L = 20.2 fb^{-1})
CMS eμ 8 TeV (L = 19.7 fb^{-1})
LHC combined* eμ 8 TeV (L = 20 fb^{-1})
ATLAS eμ* 13 TeV (L = 139 fb^{-1})
CMS eμ 13 TeV (L = 35.9 fb^{-1})
CMS τ+e/μ 13 TeV (L = 35.9 fb^{-1})
ATLAS l+jets 13 TeV (L = 139 fb^{-1})
CMS l+jets 13 TeV (L = 137 fb^{-1})
CMS all-jets* 13 TeV (L = 2.53 fb^{-1})
ATLAS all-jets 13 TeV (L = 36.1 fb^{-1})
CMS dilepton,l+jets* 13.6 TeV (L = 1.2 fb^{-1})
* Preliminary
ATLAS+CMS Preliminary
LHCtopWG
November 2022
NNLO+NNLL (pp)
NNLO+NNLL (pp̄)
Czakon, Fiedler, Mitov, PRL 110 (2013) 252004
NNPDF3.0, m_{top} = 172.5 GeV, $\alpha_s(M_Z)$ = 0.118 ± 0.001
√s [TeV]


Fiducial
LO
NLO
NNLO
ATLAS
LHC 13 TeV m_t = 172.5 GeV
Scale: $H_T/4$ PDF: NNPDF31nnlo
NNLO/NLO
NLO/LO
$\Delta\phi(\ell,\bar{\ell})/\pi$
Inclusive


ATLAS+CMS Preliminary
LHCtopWG
$\sigma_{t\bar{t}}$ summary, √s = 13 TeV
November 2022
NNLO+NNLL PRL 110 (2013) 252004
m_{top} = 172.5 GeV, $\alpha_s(M_Z)$ = 0.118±0.001
scale uncertainty
scale ⊕ PDF ⊕ α_s uncertainty
total stat
$\sigma_{t\bar{t}}$ ± (stat) ± (syst) ± (lumi)
ATLAS, dilepton eμ
EPJC 80 (2020) 528, L_{int} = 36.1 fb^{-1}
826 ± 4 ± 12 ± 16 pb
ATLAS, dilepton eμ*
ATLAS-CONF-2022-061, L_{int} = 139 fb^{-1}
836 ± 1 ± 12 ± 16 pb
ATLAS, l+jets
PLB 810 (2020) 135797, L_{int} = 139 fb^{-1}
830 ± 0.4 ± 36 ± 14 pb
ATLAS, all-jets
JHEP 01 (2021) 033, L_{int} = 36.1 fb^{-1}
864 ± 4.3 ± 126 ± 18 pb
CMS, dilepton eμ
PRL 116 (2016) 052002, L_{int} = 43 pb^{-1}, 50 ns
746 ± 58 ± 53 ± 36 pb
CMS, dilepton eμ
EPJC 79 (2019) 368, L_{int} = 35.9 fb^{-1}, 25 ns
803 ± 2 ± 25 ± 20 pb
CMS, dilepton τ+e/μ
JHEP 02 (2020) 191, L_{int} = 35.9 fb^{-1}, 25 ns
781 ± 7 ± 62 ± 20 pb
CMS, l+jets
JHEP 09 (2017) 051, L_{int} = 2.2 fb^{-1}, 25 ns
888 ± 2 ± 26 ± 20 pb
CMS, all-jets *
CMS-PAS TOP-16-013, L_{int} = 2.53 fb^{-1}, 25 ns
834 ± 25 ± 118 ± 23 pb
CMS, l+jets
PRD 104 (2021) 092013, L_{int} = 137 fb^{-1}, 25 ns
791 ± 1 ± 21 ± 14 pb
NNPDF3.0 JHEP 04 (2015) 040
MMHT14 EPJC 75 (2015) 5
CT14 PRD 93 (2016) 033006
ABM12 PRD 89 (2015) 054028
[$\alpha_s(m_Z)$ = 0.113]
* Preliminary
$\sigma_{t\bar{t}}$ [pb]

The two-loop frontier: $2 \to 3$

5-POINT 2-LOOP - MASSLESS: ALL FAMILIES

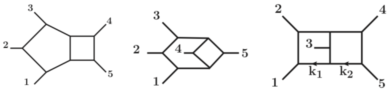
→ T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001 [erratum: Phys. Rev. Lett. **116** (2016) no.18, 189903] [arXiv:1511.05409 [hep-ph]].

→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].

→ D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang and S. Zoia, Phys. Rev. Lett. **123** (2019) no.4, 041603

→ D. Chicherin and V. Sotnikov, JHEP **20** (2020), 167

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, "Caravel: A C++ framework for the computation of multi-loop amplitudes with numerical unitarity," Comput. Phys. Commun. **267** (2021), 108069



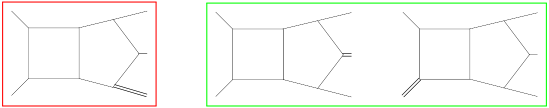
(a) penta-box (b) hexa-box (c) double-pentagon

FIG. 1: Integral topologies for massless five-particle scattering at two loops.

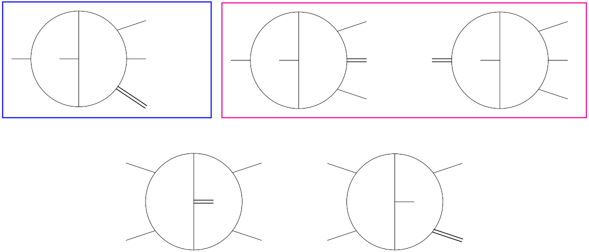
→ J. Henn, T. Peraro, Y. Xu and Y. Zhang, "A first look at the function space for planar two-loop six-particle Feynman integrals," JHEP **03** (2022), 056

5-point 2-loop - one leg off-shell: all families

→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].
→C. G. Papadopoulos and C. Wever, JHEP **2002** (2020) 112
→S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199
→ S. Abreu, H. Ita, B. Page and W. Tschernow, JHEP **03** (2022), 182 [arXiv:2107.14180 [hep-ph]].
→ A. Kardos, C. G. Papadopoulos, A. V. Smirnov, N. Syrrakos and C. Wever, [arXiv:2201.07509 [hep-ph]].



The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

NNLO 2→3

NNLO QCD: $pp \to \gamma\gamma\gamma + X$
leading-colour approximation for double-virtual

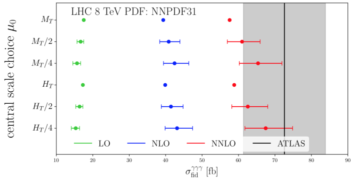


Figure 1. Predictions for the fiducial cross-section in LO (green), NLO (blue) and NNLO (red) QCD versus ATLAS data (black). Shown are predictions for six scale choices. The error bars on the theory predictions reflect scale variation only. For two of the scales only the central predictions are shown.

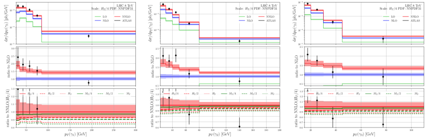


Figure 2. p_T distribution of the hardest photon γ_1 (left), γ_2 (center) and the softest one γ_3 (right). Top plot shows the absolute distribution at NNLO (red), NLO (blue) and LO (green) versus ATLAS data (black). Middle plot shows same distributions but normalized to the NLO. Bottom plot shows central NNLO predictions for 6 different scale choices (only the central scale is shown) with respect to the default choice $\mu_0 = H_T/4$. The bands represent the 7-point scale variations about the corresponding central scales.

→ H. A. Chawdhry, M. L. Czakon, A. Mitov and R. Poncelet, JHEP **2002** (2020) 057

NNLO QCD: $pp \to \gamma\gamma\gamma + X$

leading-colour approximation for double-virtual

fiducial setup for $pp \to \gamma\gamma\gamma + X$; used in the ATLAS 8 TeV analysis of Ref. [37]
$p_{T,\gamma_1} \geq 27\,\text{GeV}$, $p_{T,\gamma_2} \geq 22\,\text{GeV}$, $p_{T,\gamma_3} \geq 15\,\text{GeV}$, $0 \leq \lvert\eta_\gamma\rvert \leq 1.37$ or $1.56 \leq \lvert\eta_\gamma\rvert \leq 2.37$,
$\Delta R_{\gamma\gamma} \geq 0.45$, $m_{\gamma\gamma\gamma} \geq 50\,\text{GeV}$, Frixione isolation with $n = 1$, $\delta_0 = 0.4$, and $E_T^{\text{ref}} = 10\,\text{GeV}$.

Table 1: Definition of phase space cuts.

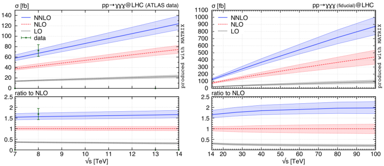


Figure 4: Fiducial cross sections for $pp \to \gamma\gamma\gamma + X$ as a function of the centre-of-mass energy at LO (black dotted), at NLO (red dashed), and at NNLO (blue, solid) The green data point at 8 TeV corresponds to the cross section measured by ATLAS in Ref. [37].

→ S. Kallweit, V. Sotnikov and M. Wiesemann, Phys. Lett. B **812** (2021) 136013

NNLO 2→3

NNLO QCD: $pp \to 3\text{jets} + X$
leading-colour approximation for double-virtual

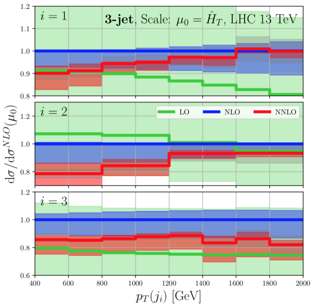


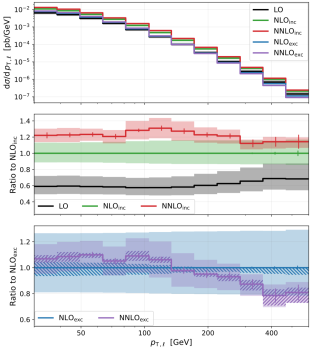
FIG. 1: The three panels show the ith leading jet transverse momentum $p_T(j_i)$ for $i = 1, 2, 3$ for the production of (at least) three jets. LO (green), NLO (blue) and NNLO (red) are shown for the central scale (solid line). 7-point scale variation is shown as a coloured band. The grey band corresponds to the uncertainty from Monte Carlo integration.

→ M. Czakon, A. Mitov and R. Poncelet, Phys. Rev. Lett. **127** (2021) no.15, 152001 [arXiv:2106.05331 [hep-ph]].

→X. Chen, T. Gehrmann, N. Glover, A. Huss and M. Marcoli, [arXiv:2203.13531 [hep-ph]].

NNLO 2→3

NNLO QCD: $pp \to Wb\bar{b} + X$
leading-colour approximation for double-virtual



→ H. B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, Phys. Rev. D **106** (2022) no.7, 074016 [arXiv:2205.01687 [hep-ph]].

NNLO 2→3

NNLO QCD: $pp \to \gamma j_1 j_2 + X$

Full-colour; sub-leading colour contributions negligible

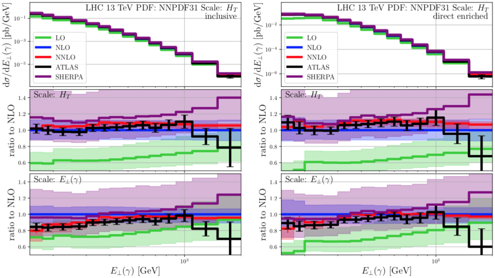


Figure 4. Differential cross sections w.r.t. the transverse energy of the photon $E_\perp(\gamma)$ in the *inclusive* (left plot) and *direct-enriched* (right plot) phase space at LO (green), NLO (blue) and NNLO (red) QCD compared to data (black) and SHERPA (purple) prediction provided by ATLAS[37]. The top panels show the absolute values for the H_T scale choice. The middle (bottom) panel shows the ratio to NLO QCD using the H_T ($E_\perp(\gamma)$) scale. The coloured bands show scale variation and the vertical coloured bars indicate statistical uncertainties.

→ S. Badger, M. Czakon, H. B. Hartanto, R. Moodie, T. Peraro, R. Poncelet and S. Zoia, [arXiv:2304.06682 [hep-ph]].

NNLO 2→3

→ V. Sotnikov, [arXiv:2207.12295 [hep-ph]].

	Comment	Complete analytic results	Public numerical code	Cross sections
$pp \to jjj$	l.c.	Abreu et al.	Abreu et al.	Chen et al., Czakon et al.
$pp \to \gamma\gamma j$	l.c.*	Agarwal et al., Chawdhry et al.	Agarwal et al.	Chawdhry et al.
$pp \to \gamma\gamma\gamma$	l.c.*	Abreu et al., Chawdhry et al.	Abreu et al.	Chawdhry et al., Kallweit et al.
$pp \to \gamma\gamma j$		Agarwal et al.		
$gg \to \gamma\gamma g$	NLO loop induced	Badger et al.	Badger et al.	Badger et al.
$pp \to Wb\bar{b}$	l.c.*, on-shell W	Badger et al.		
$pp \to W(l\nu)b\bar{b}$	l.c.	Abreu et al., Hartanto et al.		Hartanto et al.
$pp \to W(l\nu)jj$	l.c.	Abreu et al.		
$pp \to Z(l\bar{l})jj$	l.c.*	Abreu et al.		
$pp \to W(l\nu)\gamma j$	l.c.*	Badger et al.		
$pp \to Hb\bar{b}$	l.c., b-quark Yukawa	Badger et al.		

Table 1: Known two-loop QCD corrections for five-point scattering processes at hardon colliders. "l.c." refers to the calculations in the leading-color approximation; "l.c.*" means that in addition non-planar l.c. contributions are omitted. All public codes employ `PentagonFunctions++` Chicherin and Sotnikov, Chicherin et al. for numerical evaluation of special functions.

The three-loop frontier: 2$\to$2

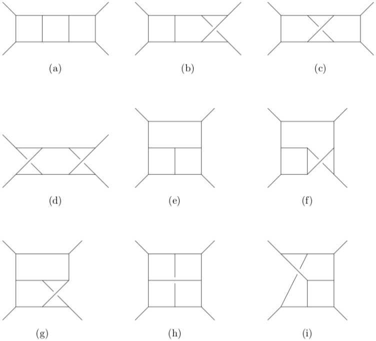


Figure 1. The nine integral families needed to describe all master integrals for three-loop massless four-particle scattering. The external legs are associated with the momenta p_1, p_3, p_4 and p_2 in clockwise order starting with the top left corner.

→ J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP **07** (2013), 128

→ J. Henn, B. Mistlberger, V. A. Smirnov and P. Wasser, JHEP **04** (2020), 167

3-LOOP CALCULATIONS

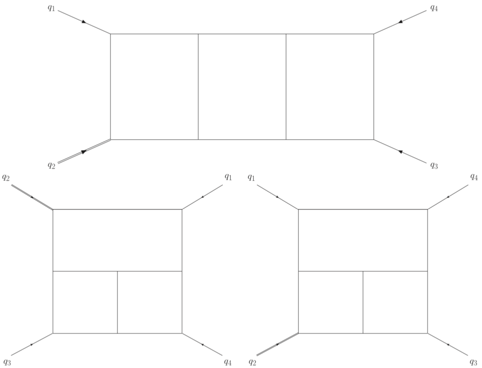


Figure 1. The F1 (top), F2 (bottom left) and F3 (bottom right) top-sector diagrams. The double line represents the massive particle and all external momenta are taken to be incoming.

→ S. Di Vita, P. Mastrolia, U. Schubert and V. Yundin, JHEP **09** (2014), 148

→ D. D. Canko and N. Syrrakos, JHEP **04** (2022), 134

3-LOOP CALCULATIONS

→ F. Caola, A. Von Manteuffel and L. Tancredi, "Diphoton Amplitudes in Three-Loop Quantum Chromodynamics," Phys. Rev. Lett. **126** (2021) no.11, 112004

→ F. Caola, A. Chakraborty, G. Gambuti, A. von Manteuffel and L. Tancredi, "Three-loop helicity amplitudes for four-quark scattering in massless QCD," JHEP **10** (2021), 206

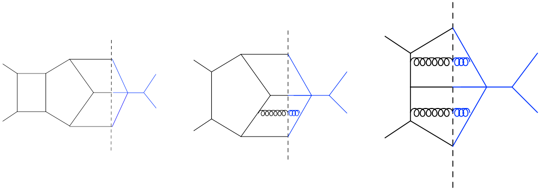
→P. Bargiela, F. Caola, A. von Manteuffel and L. Tancredi, "Three-loop helicity amplitudes for diphoton production in gluon fusion," JHEP **02** (2022), 153

→ F. Caola, A. Chakraborty, G. Gambuti, A. von Manteuffel and L. Tancredi, "Three-Loop Gluon Scattering in QCD and the Gluon Regge Trajectory," Phys. Rev. Lett. **128** (2022) no.21, 212001

→ F. Caola, A. Chakraborty, G. Gambuti, A. von Manteuffel and L. Tancredi, "Three-loop helicity amplitudes for quark-gluon scattering in QCD," [arXiv:2207.03503 [hep-ph]].

What do we need for an NNLO calculation ?

$$p_1\,,\,p_2 \to p_3,\,\ldots,\,p_{m+2}$$



What do we need for an NNLO calculation ?

$$\sigma_{NNLO} \rightarrow \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left|M_m^{(1)}\right|^2\right) J_m(\Phi) \qquad VV$$

$$+ \int_{m+1} d\Phi_{m+1} \left(2Re\left(M_{m+1}^{(0)*} M_{m+1}^{(1)}\right)\right) J_{m+1}(\Phi) \qquad RV$$

$$+ \int_{m+2} d\Phi_{m+2} \left|M_{m+2}^{(0)}\right|^2 J_{m+2}(\Phi) \qquad RR$$

$RV + RR \rightarrow$ antenna-S, colorfull-NNLO, sector-improved residue subtraction, nested soft-collinear, local analytic sector subtraction, projection to born, q_T, N-jetiness

→A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
→ P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
→ M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
→S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
→ R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002
→ M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam and G. Zanderighi, Phys. Rev. Lett. **115**, no. 8, 082002 (2015)
→ F. Caola, K. Melnikov and R. Röntsch, Eur. Phys. J. C **77**, no. 4, 248 (2017)
→ L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, arXiv:1806.09570 [hep-ph].

Amplitude reduction

OPP AT TWO LOOPS

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1,l_2;\{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1,l_2;\{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\texttt{cut equations}: D_{i_1} = D_{i_2} = \ldots = D_{i_m} = 0$$

$$\Delta_{i_1 i_2 \ldots i_m}(l_1,l_2;\{p_i\}) \rightarrow spurious \oplus ISP - irreducible\ integrals$$

OPP AT TWO LOOPS

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$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \dots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \dots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \dots D_{i_m}}$$

ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

→ P. Mastrolia, T. Peraro and A. Primo, arXiv:1605.03157 [hep-ph].

→J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

→ H. Ita, arXiv:1510.05626 [hep-th].

→ C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, Comput. Phys. Commun. **267** (2021), 108069

HELAC-2LOOP FOR AMPLITUDE CONSTRUCTION: THE ALGORITHM

→ G. Bevilacqua, D. D. Canko, A. Kardos and C. G. Papadopoulos, J. Phys. Conf. Ser. **2105** (2021) no.5, 012010

$$n\!-\!\textit{particle},\ 2\!-\!\textit{loop Amplitude} \longrightarrow (n\!+\!2)\!-\!\textit{particle},\ 1\!-\!\textit{loop Amplitude}$$

1) Definition of the flavor of the $n+1$ and $n+2$ particles.

2) Generation of the $n+2$ color-states ($(n+2)!$, Color-Flow Representation).

3) Generation of Blob-Topologies.

4) Cut of the topologies in the k_3-line (middle-line) $\rightarrow$ the 2 extra particles.

5) Flavor-Color Dressing of the 1-loop loop-particles.

6) Second cut of the blob-topology $\rightarrow$ tree-level graph ($n+4$ color-states).

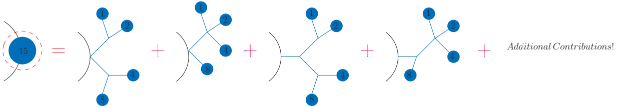
7) Creation of currents contributing to the configuration (Dyson-Schwinger to blobs).

8) Reduction of the $n+4$ color-states to n and identification of N_C power.

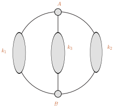
9) Storing of the numerator information to the `Skeleton`.

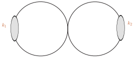
Two-loop blob-topologies

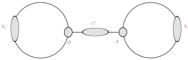
- Binary representation for the particles: e.g. for $n = 4$, $\{1,2,3,4\} \to \{1,2,4,8\}$
- What a blob and its level are?



- List representation for the 3 grand blob-topologies:

Theta−topologies:  $\equiv \{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\}$

Infinity−topologies:  $\equiv \{\{k_1\},\{k_2\}\}$

Dumbbell−topologies:  $\equiv \{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\}$

Numerators and numerics in 4 dimensions

Process	loop-flavors	Color	Size	Time	Numerators
$gg \to q\bar{q}$	$\{g, g_h, \bar{g_h}, q, \bar{q}\}$	full	16.1 MB	3m 14.509s	13856
$gg \to gg$	$\{g, g_h, \bar{g_h}\}$	leading	8.9 MB	15.017s	4560
$gg \to gg$	$\{g, g_h, \bar{g_h}, q, \bar{q}\}$	full	110.6 MB	6m 54.574s	89392
$gg \to ggg$	$\{g, g_h, \bar{g_h}\}$	leading	300.0 MB	21m 42.609s	81480

Comments on the skeletons:

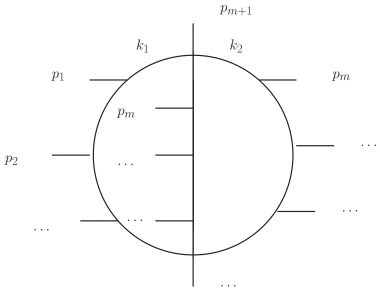
1. n increase $\longrightarrow$ complexity increase
2. leading color to full color $\longrightarrow$ complexity increase
3. Timings a bit large $\longrightarrow$ Skeleton constructed only once per process!
4. Much numerators (some are identical) $\longrightarrow$ Room for improving efficiency!

Some numerical results for numerators with gluons as external and loop particles ($h = -- \to --$)[1]:

1. $N_{\{\{1,2\},\{12\},\{\},\{\},\{\}\}} = 17052219.315419123 + 64639250.888367772i$.
2. $N_{\{\{1,2\},\{4,8\},\{\},\{\},\{\}\}} = -12231870819598.090 + 5124375444085.5430i$.
3. $N_{\{\{1,2\},\{4\},\{8\},\{\},\{\}\}} = -1268111397619.5310 + 195312105699.88257i$.
4. $N_{\{\{2,1\},\{8\},\{\},\{4\},\{\}\}} = -49731029299.352333 + 15599344.440385548i$.

- Perfect agreement in cross-checks with `FeynArts` + `FeynCalc`!

Feynman Integrals



$$\frac{\mathcal{N}\left(k_1, k_2; \{p_i\}_{i=1}^{m+1}, \{\varepsilon\}\right)}{\left(k^2 - {M_0}^2\right)\left((k_1+p_1)^2 - {M_1}^2\right)\ldots\left((k_1-k_2-p_{m+1})^2 - {M_j}^2\right)\ldots\left({k_2}^2 - {M_l}^2\right)}$$

The current approach

- m independent momenta, L loops, $N = L(L+1)/2 + Lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k_1, k_2\} + p_i)^2 - M_i^2$
- Definition
$$F[a_1, \dots, a_N] = C_L \int \frac{1}{D_1^{a_1} \dots D_N^{a_N}} \prod_{i=1}^{L} \left[d^d k_i\right]$$
with a_i being zero, positive or negative integers.
- Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: `SecDec`, Weinzierl, Anastasiou&Sterman

→ C. Anastasiou and G. Sterman, arXiv:1812.03753 [hep-ph].

→ S. Kromin, N. Schwanemann and S. Weinzierl, [arXiv:2208.01060 [hep-th]].

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→F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

→K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

IBP identities:

$$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

reduce *all* Feynman Integrals to a finite subset → Master Integrals.

$$F[a_1, \dots, a_N] = \sum_i R_i\left(\{p\}, d\right) G_i\left[a'_1, \dots, a'_N\right]$$

The current approach

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- Feynman parameters, Mellin-Barnes, Differential Equations

→ Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

→ V. A. Smirnov, Phys. Lett. B **460** (1999) 397

→ T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

→ J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Or numerical: SecDec, Weinzierl, Anastasiou&Sterman

→ C. Anastasiou and G. Sterman, arXiv:1812.03753 [hep-ph].

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→S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

→ S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

→ C. Anastasiou and G. Sterman, arXiv:1812.03753 [hep-ph].

→ S. Kromin, N. Schwanemann and S. Weinzierl, [arXiv:2208.01060 [hep-th]].

Differential Equations Approach

- The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$F[a_1,\ldots,a_N] \to G[a'_1,\ldots,a'_N]$$

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1,\ldots,a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1,\ldots,b_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1,\ldots,a'_n]$$

- Find the proper basis; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f\left(\varepsilon,\{x_i\}\right) = \varepsilon A_m\left(\{x_i\}\right) f\left(\varepsilon,\{x_i\}\right)$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

$\star$ f not MI!

→ J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

→ B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

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$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1,\ldots,a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1,\ldots,b_n] \to \sum C_{a_1',\ldots,a_n'} G[a_1',\ldots,a_n']$$

- Find the proper basis; Bring the system of equations in a form suitable to express the MI in terms of GPs

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$\star$ f not MI!

→ J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

→ B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

- Iterated Integrals →K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n,\ldots,a_1,x\right)=\int_0^x dt\,\frac{1}{t-a_n}\mathcal{G}\left(a_{n-1},\ldots,a_1,t\right)$$

→ J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005), 177

- Elliptic Integrals

→ L. Adams and S. Weinzierl, Phys. Lett. B **781** (2018), 270-278

→ J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, JHEP **01** (2019), 023

- Numerical approach [one-mass double-pentagon]
  Generalised power series expansion

→ F. Moriello, JHEP **01** (2020), 150

→ M. Hidding, Comput. Phys. Commun. **269** (2021), 108125

→ X. Liu and Y. Q. Ma, Comput. Phys. Commun. **283** (2023), 108565

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

→ A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

→C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

→C. Bogner and F. Brown

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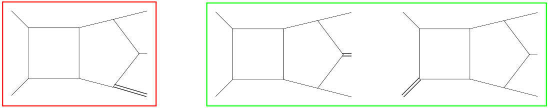
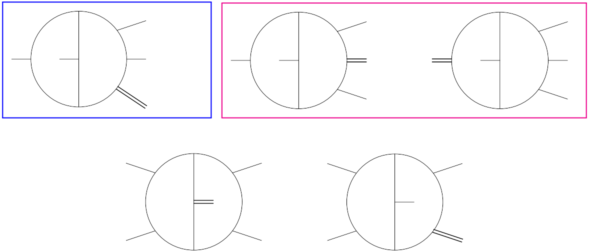
The SDE approach

5-POINT TWO-LOOP ONE-MASS



The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

Pentabox - one leg off-shell: P1

→J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601

→S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117

→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

- Also from direct differentiation of MI wrt to x. Just g in terms of FI.

$$\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \vec{g}$$

- ℓ_b, are independent of x, some depending only on the reduced invariants, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. M_b are independent of the invariants.
- number of letters smaller than in AIMPTZ representation
- Main contribution for us from AIMPTZ: the canonical basis (+ numerics)

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$$\frac{d\log\left(W_a\right)}{dx}$$

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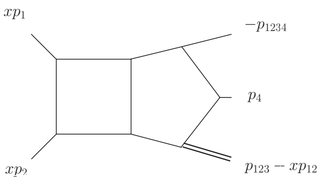
PENTABOX - ONE LEG OFF-SHELL: P1



$$q_1 \to p_{123} - xp_{12},\ q_2 \to p_4,\ q_3 \to -p_{1234},\ q_4 \to xp_1$$

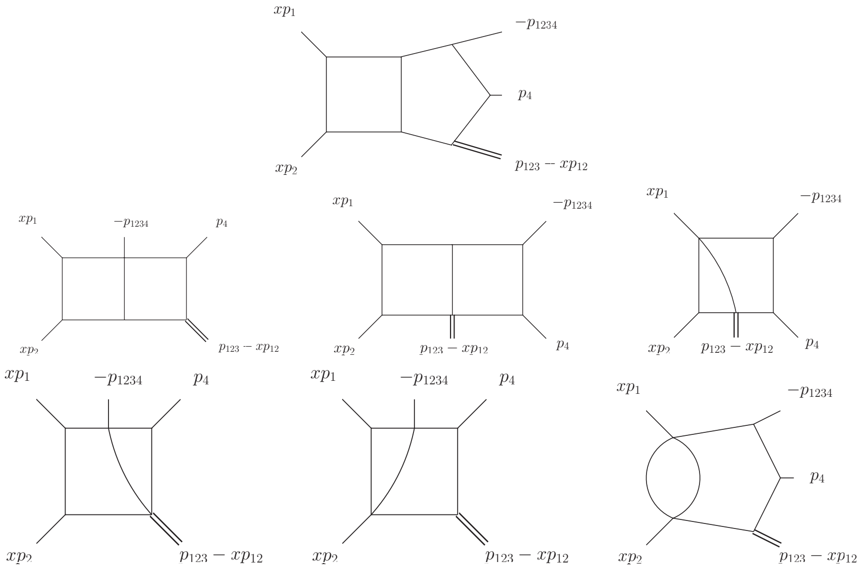
SDE parametrisation: n off-shell legs $\to$ $n-1$ off-shell legs $+$ the x variable.

→ C. G. Papadopoulos, "Simplified differential equations approach for Master Integrals," JHEP **1407** (2014) 088

- $p_i,\ i = 1\dots5$, satisfy $\sum_1^5 p_i = 0$, with $p_i^2 = 0,\ i = 1\dots5,\ p_{i\dots j} := p_i + \dots + p_j$. The set of independent invariants: $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$.

$$q_1^2 = (1-x)(S_{45} - S_{12}x),\ s_{12} = (S_{34} - S_{12}(1-x))\,x,\ s_{23} = S_{45},\ s_{34} = S_{51}x,$$
$$s_{45} = S_{12}x^2,\ s_{15} = S_{45} + (S_{23} - S_{45})x$$

PENTABOX - ONE LEG OFF-SHELL: P1



4-POINT UP TO TWO LEGS OFF-SHELL

→ J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1405** (2014) 090
→ T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs, JHEP **06** (2014), 032
→ F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1409** (2014) 043
→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072
→ T. Gehrmann, A. von Manteuffel and L. Tancredi, JHEP **09** (2015), 128

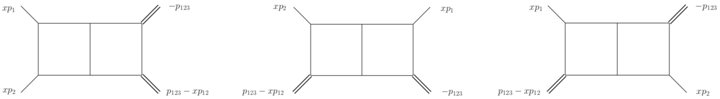


Figure 3. The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

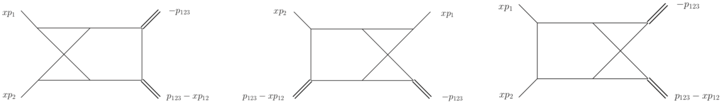


Figure 4. The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

As well as planar and nonplanar double box with one off-shell leg expressed in UT basis.

Pentabox - one leg off-shell: P1-3

$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_a \frac{1}{x - \ell_a} \mathbf{M}_a \mathbf{g}$$

PENTABOX - ONE LEG OFF-SHELL: P1-3

$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_a \frac{1}{x-\ell_a} \mathbf{M}_a \mathbf{g}$$

$$\mathbf{g} = \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right)$$
$$+ \epsilon^2 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right)$$
$$+ \epsilon^3 \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)} \right)$$
$$+ \epsilon^4 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)} \right.$$
$$\left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)} \right) + \ldots$$

$$\mathcal{G}_{ab\ldots} := \mathcal{G}(\ell_a, \ell_b, \ldots; x)$$

→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. Results.txt

PENTABOX - ONE LEG OFF-SHELL: KINEMATICAL REGIONS

- Euclidean region:

$$\left\{ S12 \to -2, S23 \to -3, S34 \to -5, S45 \to -7, S51 \to -11, x \to \frac{1}{4} \right\}$$

no letter l in the region $[0, x]$, all boundary terms real. [very fast GiNaC]

Family	W=1	W=2	W=3	W=4
P_1 (g_{72})	17 (14)	116 (95)	690 (551)	2740 (2066)
P_2 (g_{73})	25 (14)	170 (140)	1330 (1061)	4950 (3734)
P_3 (g_{84})	22 (12)	132 (90)	1196 (692)	4566 (2488)

TABLE: Number of GP entering in the solution. In parenthesis we give the corresponding number for the non-zero top-sector basis elements.

- with timings, running the GiNaC Interactive Shell ginsh, given by 1.9, 3.3, and 2 seconds for P_1, P_2 and P_3 respectively and for a precision of 32 significant digits
- A very different canonical basis, several elements start at ϵ^4.

Summary & Outlook

- Non-planar families
  - We have completed the hexa-box families, N_1, N_2, N_3.
  - Checks against known results successful.
  - Next task: double-pentagon families, N_4, N_5.
- SDE approach: all MI up to 4-point with up to 2 off shell legs and 5-point with up to one off-shell leg.
- Speed-up numerical evaluation
  - Improving GPLs analytic continuation.
  - Study letters ordering in physical regions, use different mappings and/or fibrations.
  - Combine analytics with numerics → one-dimensional integral representation
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- `HELAC2LOOP`: generic approach to amplitude reduction and evaluation

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    → A. Kardos, C. G. Papadopoulos, A. V. Smirnov, N. Syrrakos and C. Wever, JHEP **05** (2022), 033
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→ S. Abreu, H. Ita, B. Page and W. Tschernow, JHEP **03** (2022), 182

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  - SDE@1-loop → N. Syrrakos, "One-loop Feynman integrals for 2 → 3 scattering involving many scales including internal masses," JHEP **10** (2021), 041 [arXiv:2107.02106 [hep-ph]].
  - SDE@3-loop → D. D. Canko and N. Syrrakos, "Planar three-loop master integrals for 2 → 2 processes with one external massive particle," [arXiv:2112.14275 [hep-ph]].
  - UT basis determination → more criteria as experience dictates

    → H. Frellesvig and C. G. Papadopoulos, JHEP **04** (2017), 083

    → J. Henn, B. Mistlberger, V. A. Smirnov and P. Wasser, JHEP **04** (2020), 167

    → P. Wasser, "Scattering Amplitudes and Logarithmic Differential Forms,"

    → C. Dlapa, X. Li and Y. Zhang, JHEP **07** (2021), 227
  - Boundary terms determination → for UT basis elements
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Thank you for your attention !

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H.F.R.I.
Hellenic Foundation for Research & Innovation

Backup slides

Explain Theory

- The known knowns

  Newtonian gravity, Electromagnetism, QM, QFT, Einstein gravity, and a large part of the SM

  $\rightarrow$ experimental input + "perturbative" calculations
- The known unknowns

  Dark matter, dark energy, asymmetries, the rest of the SM, plus many others, such as BH, strongly interacting matter, etc.

Explain Theory

- What we can promise is to get all elements, a highly non-trivial task,
  accelerators, detectors, calculations, education, etc.
  to fully exploit the experimental data,
  so to unambiguously determine any deviation from the known knowns
- and many models (complete or incomplete) that may accommodate such deviations, i.e. discoveries.
- not excluding the unknown

$\rightarrow$ this is not different from what other scientific fields are pursuing
or the history of sciences dictates