Higher-order corrections at the LHC: current status and prospects

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Abstract. Precise theoretical predictions are necessary for revealing the nature of elementary particle interactions at the highest energies available at the LHC and the future collider FCC. I will review the scattering amplitudes computation starting with the leading-order (LO) and the next-to-leading-order (NLO) approximations. The recent progress at the two-loop frontier will be discussed along with next-to-next-to-leading-order (NNLO) results for $2 \to 3$ processes.

Keywords: scattering amplitudes, higher-order corrections, two-loop Feynman integrals, amplitude reduction at the integrand level

1 Introduction

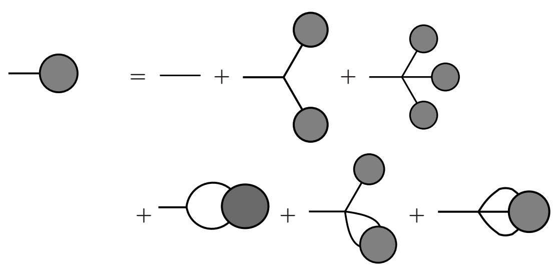
During the last decade, we have witnessed three of the most significant discoveries of all time in the field of fundamental interactions, which can be summarised in the detection of the Higgs particle [1, 2] at the LHC, the detection of the gravitational waves (GW) [3] and the event-horizon-scale image of the supermassive black hole (BH) candidate M87* [4]. These discoveries have been made possible due to very sophisticated and technically advanced facilities such as the LHC accelerator, the ATLAS and CMS detectors, the Laser Interferometer Gravitational-Wave detectors, and the Event Horizon Telescope. On the other hand, these discoveries would have been impossible without the use of very advanced calculations and simulations regarding scattering processes in the case of Higgs particle, the description of merging BH in the case of GW detection, and the accretion process in the case of BH imaging, exploiting our best possible understanding of quantum field theory, especially QCD, general relativity (GR) and BH physics. It is fair to acknowledge that we are now better positioned to improve our understanding of fundamental interactions, including electroweak, strong, and gravitational interactions.

In this presentation, I will focus on the precise theoretical predictions necessary to describe scattering processes at high-energy colliders.

2 LO

Leading-order calculations refer to tree-order Feynman graphs. In the 80's the CALKUL collaboration [5] initiated a systematic approach to multi-particle processes. With the introduction of spinor techniques [6] for helicity amplitude calculations, analytic as well as numerical computation has been significantly advanced. `MadGraph` was one of the first attempts to automatise within a computer programme the calculation of tree-order scattering amplitudes [7].

Nevertheless, calculations based on Feynman graphs suffered from the $n!$ growth of the number of individual graphs as a function of the number of particles, n, involved in the process. The first attempt to go beyond Feynman graphs was the work of Berends and Giele [8]. By setting up a recursive equation they were able to prove the surprisingly compact result of Parke and Taylor for n-gluon amplitude [9]. Later on, the recursive equations have been extended to deal with any quantum field theory [10–12]. The so-called Dyson-Schwinger recursive equations refer to an arbitrary $1 \to (n-1)$ process and to all orders in perturbation theory [13, 14], as shown below.



The first line refers to tree-order calculation (no loop involved), and it has been implemented in a numerical code, `HELAC` [12, 15]. `HELAC` is the first computational programme able to compute arbitrary $n-$particle scattering at LO. Nowadays the use of recursive equations to deal with tree-order amplitudes has become the standard procedure [16–18].

In the tables below one can see the gain in complexity achieved with the recursive equations.

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

Table 1. Number of Feynman graphs contributing to $n-$gluon process at tree order

$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

Table 2. Number of Berends-Giele currents needed to calculate $n-$gluon colour-stripped amplitude at tree order

3 NLO

At the NLO, calculations become not only more complex but more insightful from the physics point of view. It is the first time to encounter, the renormalisation of the so-call ultraviolet (UV) divergencies [19] and the application of the factorisation of long- and short-distance physics [20].

Schematically for a process at the parton level, the NLO cross section is given by

$$
\begin{aligned}
\sigma_{NLO} = & \int_m d\Phi_m^{D=4}(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*}M_m^{(CT)}(\epsilon_{UV})))J_m(\Phi) \quad \leftarrow LO \\
& + \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*}M_m^{(1)}(\epsilon_{UV},\epsilon_{IR}))J_m(\Phi) \quad \leftarrow Virtual \\
& + \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}}|M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \quad \leftarrow Real
\end{aligned}
$$

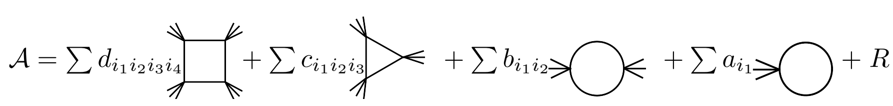
$J_m(\Phi)$ is the jet function satisfying $J_{m+1} \to J_m$, guaranteeing infrared safeness. IR and UV refer to the infrared and ultraviolet divergencies respectively. UV divergencies are cancelled by the Lagrangian counter-terms (CT) introducing the scale dependence on the renormalisation scale μ_R. IR divergencies are cancelled by collinear counter-terms that are absorbed in the parton distribution functions introducing the QCD factorization scale μ_F.

The general form of the one-loop amplitude can be written as follows:

$$
\mathcal{A} \quad = \sum_{I\subset\{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)}d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i\in I}\bar{D}_i(\bar{q})}
$$

where $\bar{D}_i(\bar{q})$ represent inverse Feynman propagators and $\bar{N}_I(\bar{q})$ is the numerator function, with all momenta in $d-$dimensions.

The amplitude can be written in terms of known Feynman Integrals [21] using either tensor-reduction [22] or unitarity methods [23].



where R denotes collectively the so-called *rational terms*.

In contrast with the above-mentioned methods that work at the *integral level*, OPP [24] for the first time introduced the reduction at the *integrand level*, as shown in the following decomposition

$$
\begin{aligned}
N(q) = &\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0i_1i_2i_3) + \tilde{d}(q;i_0i_1i_2i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0i_1i_2) + \tilde{c}(q;i_0i_1i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0i_1) + \tilde{b}(q;i_0i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q;i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

by introducing the so-called spurious terms, $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$, that vanish upon integration. OPP has been implemented in `CutTools` [25]. On the other hand, using Dyson-Schwinger equations applied to every one-loop topology, `HELAC-1LOOP` [26], based on the OPP method, was the first computational framework to numerically evaluate the one-loop amplitude for an arbitrary scattering process. Combined with the `HELAC-DIPOLES` [27] that addresses the one-parton unresolved contribution (so-called real corrections) to the NLO cross-section, the `HELAC-NLO` [28] was the first computational framework to numerically evaluate NLO cross sections for an arbitrary scattering process. It is fair to say that the so-called NLO-revolution [29] provided the necessary tools [30–33, 18] for the physics analysis at the LHC. It is worthwhile to mention that calculations with eight particles attached to the loop have been recently completed [34], based on `HELAC-NLO`.

4 NNLO

NNLO corrections for $2 \to 3$ processes is the current frontier to provide precise theoretical calculations to the LHC experiments [35]. The NNLO cross section at the parton level can be written as follows:

$$
\begin{aligned}
\sigma_{NNLO} = \sigma_{NLO} & \\
&+ \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left|M_m^{(1)}\right|^2\right) J_m(\Phi) \qquad VV \\
&+ \int_{m+1} d\Phi_{m+1} \left(2Re\left(M_{m+1}^{(0)*} M_{m+1}^{(1)}\right)\right) J_{m+1}(\Phi) \qquad RV \\
&+ \int_{m+2} d\Phi_{m+2} \left|M_{m+2}^{(0)}\right|^2 J_{m+2}(\Phi) \qquad RR
\end{aligned}
$$

The VV (virtual-virtual) part includes the two-loop amplitude $M_m^{(2)}$. $RV + RR$ terms (real-virtual and real-real), address the one-loop one-particle-unresolved and the tree-order two-particle-unresolved contributions, respectively. A plethora of methods and computer programmes have been developed to deal with $RV + RR$ terms, including `antenna-S` [36, 37], `colorfull-NNLO` [38], `sector-improved residue subtraction` [39], `nested soft-collinear` [40], `local analytic sector subtraction` [41], `projection to born` [42], q_T-subtraction [43] and N-jetiness [44].

Besides non-trivial performance issues regarding the immense computer resources necessary for the numerical evaluation of $RV + RR$ terms [45], it is fair to say that the main bottleneck for NNLO calculations is the computation of two-loop amplitudes (VV terms). In fact, due to the advancement of two-loop amplitude and FI computations over the last few years, NNLO QCD calculations for a few $2 \to 3$ processes have been completed, including $pp \to \gamma\gamma\gamma + X$ [46, 47], $pp \to 3$ jets $+ X$ [48, 49] and $pp \to Wb\bar{b} + X$ [50], with the last one being the only NNLO calculation with one external massive leg.

In contrast to the one-loop case, the computation of the two-loop amplitude necessitates Feynman Integrals which are not known analytically yet whereas the automation of the integral-level amplitude reduction is limited for the moment to $2 \to 2$ and $2 \to 3$ processes with gluons and quarks [51].

4.1 Two-loop Feynman Integrals

Beyond one loop, integration-by-part-identities (IBPI) [52, 53] are indispensable for the calculation of Feynman Integrals. Multi-loop FI can be defined as

$$F[a_1, \dots, a_N] = C_L \int \frac{1}{D_1^{a_1} \dots D_N^{a_N}} \prod_{i=1}^{L} \left[d^d k_i\right]$$

with a_i being zero, positive or negative integers and C_L a proper normalisation. In writing the above equation we assume that the family of FIs depend on m independent external momenta and therefore the number of independent scalar products between external and loop-momenta is given by $N = L(L+1)/2 + Lm$. Inverse propagators, $D_1 \dots D_N,\ D_i = \left(\{k_1, k_2\} + p_i\right)^2 - M_i^2$ form a basis that allows to express all N scalar products.

IBP identities express the fact that the total derivative of a regulated FI vanishes, which in the case of two loops can be written as,

$$\int d^d k d^d l \quad \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

IBPI equations reduce *all* FI to a finite subset of Master Integrals (MI), G,

$$F[a_1, \dots, a_N] = \sum_i R_i \left(\{p\}, d\right) G_i \left[{a'}_1, \dots, {a'}_N\right]$$

where $R_i \left(\{p\}, d\right)$ are rational coefficients of the kinematical invariants formed by the external momenta and the dimension d. Now the problem has been reduced in calculating a finite set of MI.

MI can be evaluated by Feynman parameters using the sector decomposition method [54–56], or Mellin-Barnes [57] representation. Nevertheless, the Differential Equations (DE) method [58] proved the most efficient one in obtaining analytic or semi-analytic results for multi-loop FI. Since the MI is a function of external momenta, one can set up differential equations by differentiating and using IBPI to bring FI to MI.

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1, \ldots, b_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

By using a proper basis of MI, one can bring the system of equations in a so-called canonical form [59],

$$\begin{array}{l} \partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\}) \\ \partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0 \end{array}$$

This allows to express the result in terms of iterated integrals [60] and in many cases of interest in terms of a particular class of functions, the Goncharov Polylogarithms [61]

$$\mathcal{G}(a_n, \ldots, a_1; x) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}(a_{n-1}, \ldots, a_1, t)$$

Boundary conditions for the system of DE can be obtained either by the expansion by regions method [62] or regularity conditions.

Fully numerical approaches to solve the system of DE have also been developed [63, 64] over the last few years.

Five-point two-loop integrals with one off-shell leg are the current frontier. In Figs (1) and (2), the top-sector representative integral is presented.

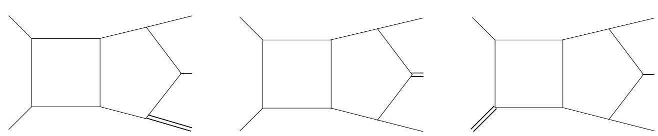


Fig. 1. The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

The P_1 family has been computed analytically in terms of GP already in 2015 [65]. The full set of planar families has been computed analytically [66] and numerically [67] based on generalised power series expansion [68], in 2020. The non-planar family N_1 has been computed in terms of GP whereas N_2, N_3 in terms of GP including one-dimensional integral representations [69]. N_1, N_2, N_3 have also been computed numerically [70] based on generalised power series expansion.

In 2014, I introduced the so-called Simplified Differential Equations approach (SDE) [71]. Based on SDE we have been able to compute all MI up to five-point two-loop integrals with one off-shell leg. To describe briefly the SDE approach, let us denote by $\mathbf{g}$ the set of MI, W_a the so-called letters in the canonical form which

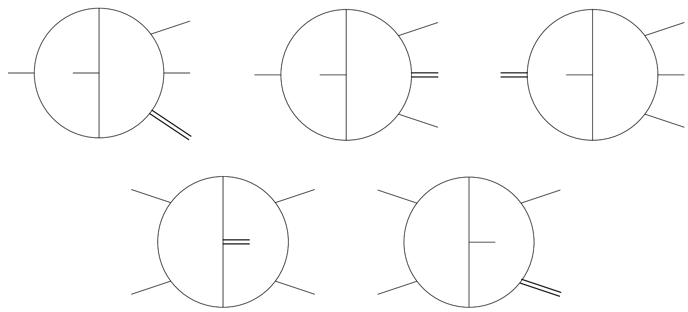


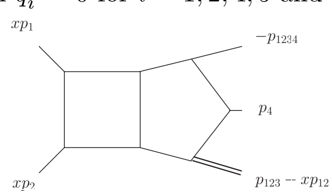
Fig. 2. The five non-planar families with one external massive leg. In the upper line the hexabox families N_1, N_2, N_3 are shown. In the lower line the double-pentagon families N_4, N_5 are shown.

are functions of the external momenta and $\tilde{M}_a$ the matrices whose elements are rational numbers.

$$d\mathbf{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \mathbf{g}$$

The canonical DE above is still a multi-variate equation with the W_a functions being algebraic functions of the kinematical invariants in general, and as such it is not straightforward to express the result in terms of GPs.

The SDE approach provides a factorisation of the canonical form, in terms of a univariate DE that is straightforwardly solvable in terms of GPs, in most cases. More specifically it accounts for the parametrisation of the external momenta, so that n momenta, with at least one of them being off-shell, are mapped to n momenta with one less off-shell momentum plus the x variable. For instance in the P_1 planar family with $q_i^2 = 0$ for $i = 1, 2, 4, 5$ and $q_3^2 \neq 0$,



the mapping $\{q_1, q_2, q_3, q_4, q_5\} \to \{p_1, p_2, p_3, p_4, p_5, x\}$ defined by (all $p_i^2 = 0$)

$$q_1 \to xp_1,\ q_2 \to xp_2,\ q_3 \to p_1 + p_2 + p_3 - x(p_1 + p_2),\ q_4 \to p_4,\ q_5 \to p_5$$

explicitly separates or factorises the dependence on x. In fact, by expressing the initial invariants $q_i \cdot q_j$ in terms of the new invariants $p_i \cdot p_j$ and the variable x, and then differentiating with respect to x the DE takes the form

$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \mathbf{g}$$

where ℓ_b depends on the new invariants $p_i \cdot p_j$, but are independent of x and M_b are just matrices of rational numbers. The fact that the DE has now simple poles in the variable x allows for a straightforward expression of its solution in terms of GP

$$\begin{aligned}
\mathbf{g} &= \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \epsilon^2\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \\
&+ \epsilon^3\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)}\right) \\
&+ \epsilon^4\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)}\right. \\
&\left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)}\right) + \ldots \\
\mathcal{G}_{ab\ldots} &:= \mathcal{G}(\ell_a, \ell_b, \ldots; x)
\end{aligned}$$

where $\mathbf{b}_0^{(i)}$ are the boundary constants[1].

4.2 Two-loop Amplitude Reduction

A generic two-loop amplitude can be decomposed following an OPP-type approach

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,n_p)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \tag{1}$$

where $S_{m;n}$ denotes the subsets of m out of n elements, and $n_p = 8$ or $n_p = 11$ depending if the reduction is performed in 4 or in d dimensions. The residues $\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)$ are composed by terms corresponding to irreducible integrals and spurious terms that vanish upon integration. As usual, the determination of the residue terms proceeds through the cut equations $D_{i_1} = D_{i_2} = \ldots = D_{i_m} = 0$. Irreducible integrals are expressed in terms of MI through IBPI.

The left-hand side of Eq.(1) is evaluated by `HELAC-2LOOP`. This is achieved by expressing the $n-$particle $2-$loop amplitude in terms on $(n+2)-$particle 1-loop amplitude and then follow an appropriately adapted `HELAC-1LOOP` construction. As in the one-loop case, a skeleton is constructed that includes all numerators for a given process. For details please refer to [72].

In the table below we present the first results on the construction of the skeleton for several processes, in both leading-colour and full-colour approximations.

[1] See references [66, 69] for details

Process	#	***Loop-Flavors***	***Color***	***Size***	***Time***	***Num/s***
$gg \to gg$	2	$\{g, c, \bar{c}\}$	Lead.	8.9 MB	15.017s	4560
$gg \to gg$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	110.6 MB	6m 54.574s	89392
$gg \to q\bar{q}$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	16.1 MB	3m 14.509s	13856
$gg \to ggg$	2	$\{g, c, \bar{c}\}$	Lead.	300.0 MB	21m 42.609s	81480

Table 3. Skeleton data for several processes (***Process***), including the flavour of the particles encountered in the loop (***Loop-Flavors***), the treatment of the colour (***Color***), the disk size of the skeleton (***Size***), the time for its creation (on a single laptop) (***Time***) and the number of numerators computed (***Num/s***).

5 Summary & Outlook

In the last few years, significant progress on the two-loop frontier has been achieved. In this presentation, I have reviewed most of the major accomplishments of this progress. For two-loop MI the current frontier is five-point two-loop integral families with up to one massive leg. Based on their analytic representation numerical tools have been developed [73–75]. With the completion of the `HELAC-2LOOP` framework, we expect that the numerical evaluation of two-loop amplitudes with up to five external particles and with up to one external off-shell momentum will be available. The successful application of the SDE approach to MI families with internal masses [76, 77] paves the road for new results, significantly widening the range of two-loop amplitudes to be computed. Based on all these developments, NNLO QCD results are expected to be copiously produced in the near future.

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