# “The Bootstrap” Program

## for integrable QFT in 1+1 dim

### S-Martrix + Form Factors

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Yerevan, May 2003

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  - Quantum field operator equations

# The "Bootstrap Program"

Construct a quantum field theory explicitly by:

1. **Calculate the 2-particle S-matrix** using
   (a) general Properties: unitarity, crossing etc.
   (b) symmetry properties
   (c) "bound state bootstrap"
   (d) "Yang-Baxter Equation"
   (e) 'maximal analyticity'

2. **Calculate the "form factors"**

$$\langle\, 0 \,|\, \phi(x) \,|\, p_1, \ldots, p_n \,\rangle^{in} = e^{-ix(p_1+\ldots+p_n)}\, f^{\phi}\,(\theta_1, \ldots, \theta_n)$$

   using
   (a) the S-matrix
   (b) LSZ-assumptions
   (c) 'maximal analyticity'

3. **Calculate the "Wightman functions"**

$$\langle\, 0 \,|\, \phi(x)\phi(y) \,|\, 0 \,\rangle = \sum_n \int \langle\, 0 \,|\, \phi(x) \,|\, n \,\rangle^{in\ in}\langle\, n \,|\, \phi(y) \,|\, 0 \,\rangle$$

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The bootstrap program classifies integrable quantum field theoretic models

# Integrability

Integrability in (quantum) field theories means:
there exits $\infty$-many conservation laws

$$\partial_\mu J_L^\mu(t,x) = 0 \quad (L = \pm 1, \pm 3, \ldots)$$

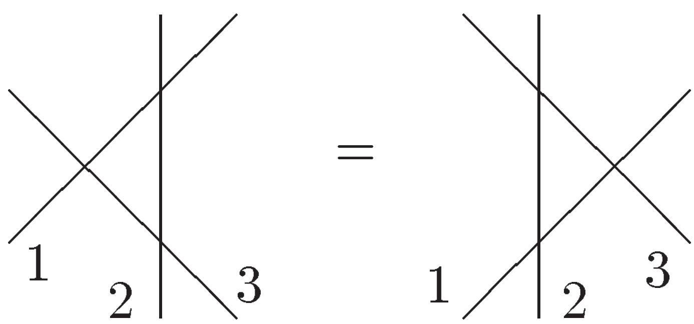
Consequence in 1+1 dim.:
The n-particle S-matrix is a product of 2-particle S-matrices

$$S^{(n)}(p_1, \ldots, p_n) = \prod_{i<j} S_{ij}(p_i, p_j)$$

In particular for the 3-particle S-matrix

$$\mathbf{S}^{(3)} = \mathbf{S}_{12}\mathbf{S}_{13}\mathbf{S}_{23} = \mathbf{S}_{23}\mathbf{S}_{13}\mathbf{S}_{12}$$

## "Yang-Baxter Equation"



# Examples:

## The sine-Gordon model

$$\ddot{\varphi}(t,x) - \varphi''(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0$$

## The Massive Thirring model

$$\mathcal{L}^{MT} = \bar{\psi}(i\gamma\partial - m)\psi - \tfrac{1}{2}g\;\bar{\psi}\gamma^{\mu}\psi\,\bar{\psi}\gamma_{\mu}\psi$$

Coleman: Both models are equivalent on the quantum level

In the following explicit solutions are given for the sine-Gordon alias massive Thirring model

## Further models

- Scaling $Z_N$-Ising models
- Nonlinear $\sigma$-models
- Gross-Neveu models
- Toda models

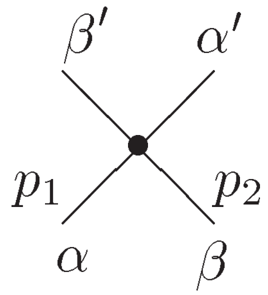
# The S-matrix

Example: Sine-Gordon alias massive Thirring model

Particles: soliton, anti-soliton and some breathers (as soliton anti-soliton bound states)

The 2-particle S-matrix:

$$S_{\alpha\beta}^{\beta'\alpha'}(p_1,p_2) =$$




$$S(\theta) = \begin{pmatrix} S_{ss} & & & & & & \\ & T_{s\bar{s}} & R_{s\bar{s}} & & & & \\ & R_{s\bar{s}} & T_{s\bar{s}} & & & & \\ & & & S_{\bar{s}\bar{s}} & & & \\ & & & & S_{sb} & & \\ & & & & & S_{bb} & \\ & & & & & & \ddots \end{pmatrix}$$

$s =$ soliton, $\bar{s} =$ anti-soliton

The rapidity difference $\theta = |\theta_1 - \theta_2|$ is defined by

$$p_i^\mu = m_i \begin{pmatrix} \cosh\theta_i \\ \sinh\theta_i \end{pmatrix}$$

## Unitarity + 'Crossing' + Yang-Baxter $\Longrightarrow$

The 'minimal' general solution of these equations
[Karowski Thun Truong Weisz 1977]

$$S_{ss}(\theta,\nu) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{t}{2}(1-\nu)}{\sinh \frac{t}{2}\nu \, \cosh \frac{t}{2}} \sinh t \frac{\theta}{i\pi}$$

where $\nu$ is a free parameter related to the couplings

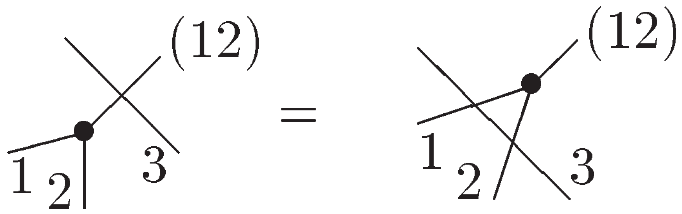
$$\frac{1}{\nu} = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi}$$

This S-matrix has been obtained before [Zamolodchikov 1977] by other methods

# The bound state S-matrix

The scattering of the bound state (12) with 3 is obtained by the "bootstrap" equation

$$S_{(12)3}(\theta_{(12)3})\,\Gamma^{(12)}_{12} \quad = \quad \Gamma^{(12)}_{12}\,S_{13}(\theta_{13})S_{23}(\theta_{23})$$




The 'bound state intertwiner' $\Gamma^{\gamma}_{\alpha\beta}$ is given by

$$i \operatorname*{Res}_{\theta=ia} \dot{S}^{\beta'\alpha'}_{\alpha\beta}(\theta) = \Gamma^{\beta'\alpha'}_{\gamma}\,\Gamma^{\gamma}_{\alpha\beta}$$

The position of the pole is related to the bound state mass

$$m_{(12)} = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos a}\ , \quad (0 < a < \pi)$$

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**Example:** The sine-Gordon 2-breather S-matrix

$$S_{bb}(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}$$

[Karowski Thun 1977]

# Form factors

Definition: Let $\mathcal{O}(x)$ be a local operator

$$\langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, \ldots, p_n \,\rangle^{in}_{\alpha_1 \ldots \alpha_n} = f^{\mathcal{O}}_{\alpha_1 \ldots \alpha_n}(\theta_1, \ldots, \theta_n)$$

$=$ 

for $\theta_1 > \ldots > \theta_n$ (elsewhere by analytic continuation)

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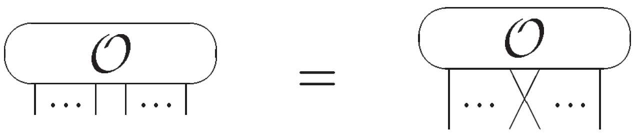
$\left.\begin{array}{r}\text{LSZ-assumptions} \\ +\ \text{'maximal analyticity'}\end{array}\right\} \Longrightarrow$ Properties of form factors

(i) Watson's equation

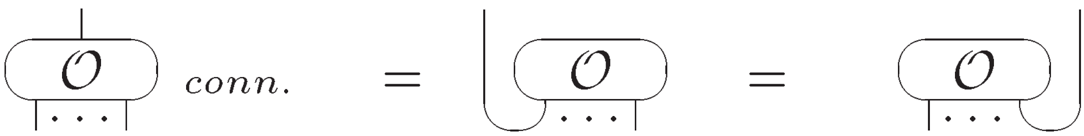
$$f^{\mathcal{O}}_{\ldots ij \ldots}(\ldots, \theta_i, \theta_j, \ldots) = f^{\mathcal{O}}_{\ldots ji \ldots}(\ldots, \theta_j, \theta_i, \ldots)\, S_{ij}(\theta_i - \theta_j)$$



(ii) Crossing

$${}_{\bar{\alpha}_1}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, \ldots, p_n \,\rangle^{in,conn.}_{\ldots \alpha_n} =$$

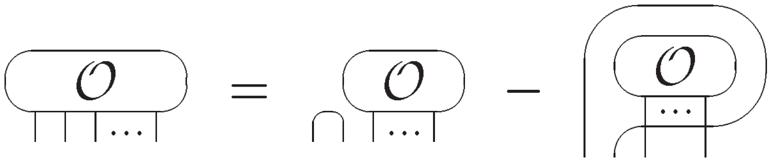
$$f^{\mathcal{O}}_{\alpha_1 \ldots \alpha_n}(\theta_1 + i\pi, \ldots, \theta_n) = f^{\mathcal{O}}_{\ldots \alpha_n \alpha_1}(\ldots, \theta_n, \theta_1 - i\pi)$$



## (iii) Annihilation recursion relation

$$\frac{1}{2i}\operatorname*{Res}_{\theta_{12}=i\pi} f^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots) = \mathbf{C}_{12}\, f^{\mathcal{O}}_{3\ldots n}(\theta_3,\ldots)\,(\mathbf{1} - S_{2n}\ldots S_{23})$$

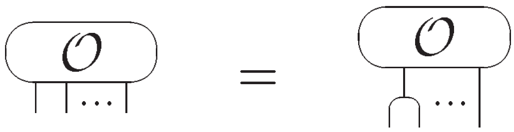
$$\frac{1}{2i}\operatorname*{Res}_{\theta_{12}=i\pi}$$




## (iv) Bound state form factors

$$\frac{1}{\sqrt{2}}\operatorname*{Res}_{\theta_{12}=ia} f^{\mathcal{O}}_{123\ldots n}(\underline{\theta}) \quad = \quad f^{\mathcal{O}}_{(12)3\ldots n}(\theta_{(12)},\underline{\theta}')\,\Gamma^{(12)}_{12}$$

$$\frac{1}{\sqrt{2}}\operatorname*{Res}_{\theta_{12}=ia}$$




## (v) Lorentz invariance

$$f^{\mathcal{O}}_{1\ldots n}(\theta_1+u,\ldots,\theta_n+u) = e^{su}\, f^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n)$$

where $s$ is the "spin" of $\mathcal{O}$

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[Karowski Weisz 1978] [Berg Karowski Weisz 1979]

[Smirnov 1992]

[Babujian Fring Karowski Zapletal 1998]

# Locality

**Theorem:** (i) – (iv) and a general crossing formula $\Rightarrow$

$$^{in}\langle\, \phi \,|\; [\mathcal{O}(x), \mathcal{O}(y)] \;|\, \psi \,\rangle^{in} = 0$$

for all matrix elements, if $x - y$ is space like.

**Proof:**

For the case of no bound states: [Smirnov 92], [Lashkevich 94]

General case: [T. Quella 99]

# Solutions of the form factor eqs. (i) – (v)

## 2-particle form factors

$$\langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \rangle^{in/out} = F\left((p_1 + p_2)^2 \pm i\varepsilon\right) = F\left(\pm\theta_{12}\right)$$

where $p_1 p_2 = m^2 \cosh\theta_{12}$.

Particle number conservation and Crossing $\Longrightarrow$

Watson's equations

$$\begin{aligned} F\left(\theta\right) &= F\left(-\theta\right) S\left(\theta\right) \\ F\left(i\pi - \theta\right) &= F\left(i\pi + \theta\right) \end{aligned}$$

"maximal analyticity" $\Rightarrow$ unique solution

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Example: The sine-Gordon 2-soliton form factor

$$F(i\pi - \theta) = \cosh\tfrac{1}{2}\theta \,\exp \int_0^\infty \frac{dt}{t} \frac{\sinh\frac{t}{2}(1-\nu)}{\sinh\frac{t}{2}\nu\,\cosh\frac{t}{2}\sinh t} \sin^2 t\frac{\theta}{2\pi}$$

[Karowski Weisz 1978]

# The General solution for $f^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1,\ldots,\theta_n)$

($\alpha_i =$ soliton, anti-soliton)

Babujian Fring Karowski Zapletal Nucl. Phys. B538 [FS] (1999) 535-586
Babujian Karowski, Nucl. Phys. B620 [FS] (2002) 407-455

**Theorem:**

$$f^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\, \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z})$$

satisfies the properties (i) – (v),

$$h(\underline{\theta},\underline{z}) = \prod_{1\le i<j\le n} F(\theta_{ij}) \prod_{i=1}^{n}\prod_{j=1}^{m} \phi(\theta_i - z_j) \prod_{1\le i<j\le m} \tau(z_i - z_j)$$

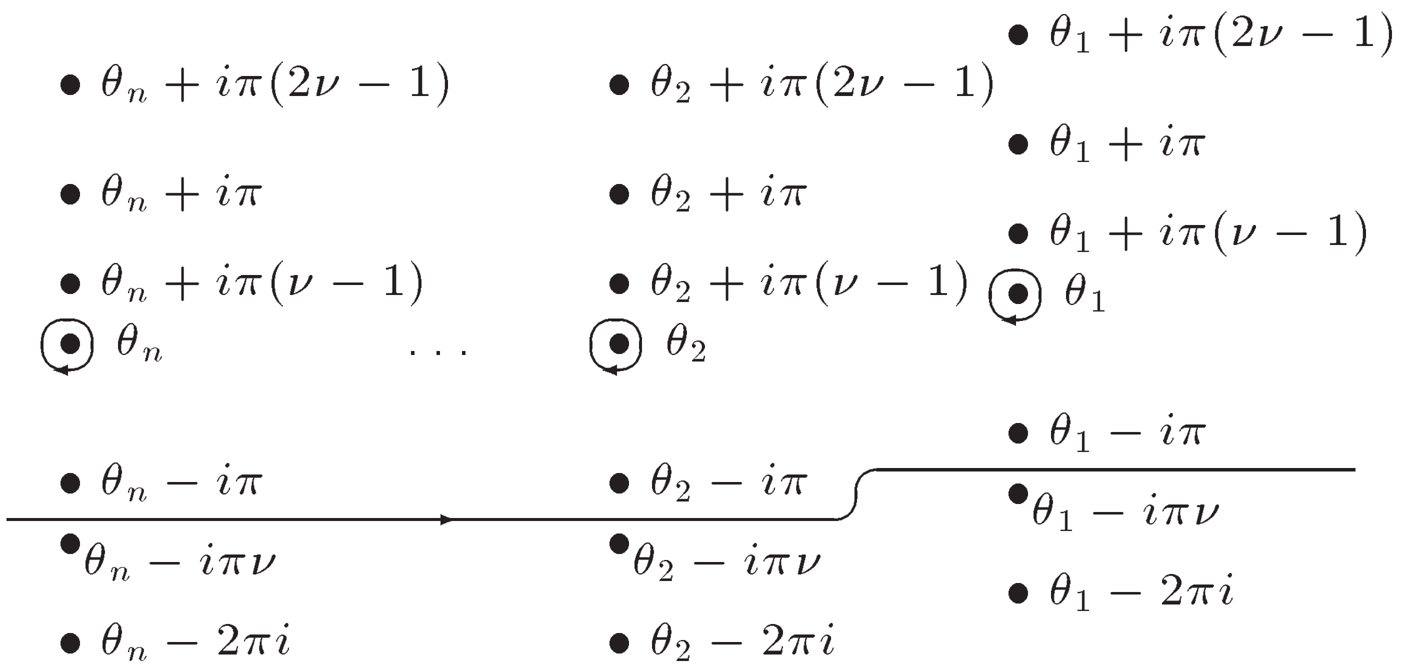
$$\tau(z) = \frac{1}{\phi(z)\,\phi(-z)} \quad , \quad \phi(z) = \frac{1}{F(z)\,F(z+i\pi)}$$

$F(z)$ is the soliton-soliton form factor

A similar formula has been obtained before by Smirnov [World Scientific 1992]

$$\left\{ \begin{array}{l} \text{integration contour } \mathcal{C}_{\underline{\theta}}, \\ \text{“Bethe ansatz” state } \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z}), \\ \text{p-function } p^{\mathcal{O}}(\underline{\theta},\underline{z}) \end{array} \right.$$

# The integration contour $\mathcal{C}_{\underline{\theta}}$

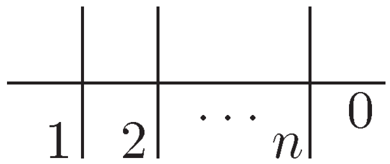


## “Off-shell Bethe ansatz” state : $\Psi_{\alpha_1\dots\alpha_n}(\underline{\theta},\underline{z})$

Define the ‘monodromy matrix’

$$T_{1\dots n,0}(\underline{\theta},\theta_0) = \dot{S}_{10}(\theta_1-\theta_0)\,\dot{S}_{20}(\theta_2-\theta_0)\cdots\dot{S}_{n0}(\theta_n-\theta_0)$$

$=$




acting in $V^{1\dots n}\otimes V_0$ with $V^{1\dots n}=V_1\otimes\cdots\otimes V_n$
(all $V_i\cong\mathbf{C}^2$ for $sl_q(2)$).

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Define the sub-matrices $A,B,C,D$ by

$$T_{1\dots n,0}(\underline{\theta},z)\equiv\begin{pmatrix} A_{1\dots n}(\underline{\theta},z) & B_{1\dots n}(\underline{\theta},z) \\ C_{1\dots n}(\underline{\theta},z) & D_{1\dots n}(\underline{\theta},z)\end{pmatrix}.$$

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The “pseudo-vacuum” consisting only of solitons

$$\Omega_{1\dots n}=s\otimes\cdots\otimes s$$

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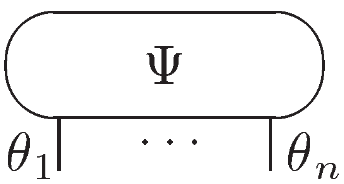
A Bethe ansatz co-vector in $V_{1\dots n}$ is defined by

$$\Psi_{1\dots n}(\underline{\theta},\underline{z}) = \Omega_{1\dots n}C_{1\dots n}(\underline{\theta},z_1)\cdots C_{1\dots n}(\underline{\theta},z_m)$$

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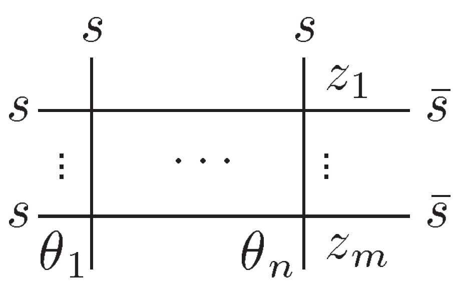

The p-function depends on the operator $\mathcal{O}(x)$.

$p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ is a polynomial in $\exp \pm z_j \ (j = 1, \ldots, m)$

The following conditions guaranty the properties (i) – (v) of the form factors.

a) $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ symmetric with respect to the $\theta$'s and the $z$'s.

b) $p_n^{\mathcal{O}}(\ldots, \theta_n + 2\pi i, \underline{z}) = \sigma_{\mathcal{O}n}\, p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$

c) $p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_n) = p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(1)}(\underline{\theta})$

$p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_1) = \sigma_{\mathcal{O}1}\, p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(2)}(\underline{\theta})$

d) $p_n^{\mathcal{O}}(\underline{\theta} + \mu, \underline{z} + \mu) = e^{s\mu}\, p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$

## Examples: (for S-G- alias M-T-Model)

Operators and their p-functions (up to normalizations)

$$\mathcal{O}(x) \leftrightarrow p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$$

### 1. The fundamental fermi field

$$\psi(x) \leftrightarrow \exp \pm \left( \sum_{j=1}^{m} z_j - \tfrac{1}{2} \sum_{i=1}^{n} \theta_i \right)$$

### 2. The current $j^\mu = \overline{\psi}\gamma^\mu\psi(x)$

$$\overline{\psi}\gamma^\pm\psi(x) \leftrightarrow \frac{\pm 1}{\sum e^{\mp\theta_i}} \left( \sum e^{-\theta_i} \sum e^{z_j} + \sum e^{\theta_i} \sum e^{-z_j} \right)$$

### 3. The $\infty$-many conserved currents

$$J_L^\pm \leftrightarrow \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{j=1}^{m} e^{L z_j} \quad (L = \pm 1, \pm 3, \ldots)$$

The higher charges satisfy the eigenvalue equation

$$\left( \int dx J_L^0(x) - \sum_{i=1}^{n} \left(p_i^+\right)^L \right) |\, p_1, \ldots, p_n\rangle^{in} = 0.$$

# The n-particle sine-Gordon breather form factors

Smirnov, World Scientific 1992
Fring Mussardo Simonetti, Nucl. Phys. B393 (1993) 413
Brazhnikov Lukyanov, Nucl. Phys. B512 (1998) 616
Babujian Karowski, Phys. Lett. B411 (1999) 53
Babujian Karowski, J. Phys. A 35 (2002) 9081-9104

$$\langle\, 0 \,|\, \mathcal{O}(0)\, |p_1, \ldots, p_n\,\rangle^{in} = K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij})$$

where the 'K-function' is

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1} \ldots \sum_{l_n=0}^{1} (-1)^{\sum l_i} \prod_{i<j} \left(1 + (l_i - l_j)\frac{i \sin \pi\nu}{\sinh \theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta}, \underline{l})$$

**Examples:**

**1.** normal ordered exponential of the field

$$\mathcal{O}(x) = :e^{i\gamma\varphi(x)}: \quad \leftrightarrow p(\underline{\theta}, \underline{l}) = \left(\frac{2}{F(i\pi)\sin \pi\nu}\right)^{\frac{n}{2}} \prod_{i=1}^{n} e^{i\pi\nu\frac{\gamma}{\beta}(-1)^{l_i}}$$

**2.** higher conserved currents
(which are typical for integrable quantum field theories)

$$J_L^{\pm}(x) \leftrightarrow \pm N_n^{(J_L)} \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{i=1}^{n} e^{L\left(\theta_i + \frac{i\pi}{2}(-1)^{l_i}\nu\right)}$$

# Quantum field operator equations

Smirnov, Nucl. Phys. B337 (1990) 156

Fring Mussardo Simonetti, Nucl. Phys. B393 (1993) 413

Mussardo Simonetti, Int. J. Mod. Phys. A9 (1994) 3307

Babujian Karowski, Phys. Lett. B411 (1999) 53

Castro-Alvaredo Fring Nucl. Phys. B618 (2001) 437

The following operator equations are equations of all their matrix elements:

## The quantum sine-Gordon equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta} : \sin\beta\varphi : (t,x) = 0$$

the "bare" mass $\sqrt{\alpha}$ is related to the renormalized mass by

$$\alpha = m^2 \frac{\pi\nu}{\sin\pi\nu} \qquad \nu = \frac{\beta^2}{8\pi - \beta^2}$$

where $m$ is the physical mass of the fundamental boson.
see also: Fateev, Phys. Lett. B324 (1994) 45,
Al. Zamolodchikov, Int. J. Mod. Phys. A10 (1995) 1125

The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction.

For fixed bare mass square $\alpha$ the physical mass goes to zero for $\nu \to 1, 2, 3, \ldots$ [Korepin's points]

At these values of the coupling the model is related to Baxters RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$

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## The trace of the energy momentum tensor

$$T^{\mu}{}_{\mu}(t,x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(:\cos\beta\varphi:(t,x) - 1)$$

Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$.

Obviously for fixed bare mass square $\alpha$ and $\beta^2 \to 8\pi$ the model will be conformal invariant which is related to a Berezinski-Kosterlitz-Thouless phase transition

## Some References:

S-matrix:


A.B. Zamolodchikov, JEPT Lett. 25 (1977) 468

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H. Babujian and M. Karowski Phys. Lett. B411 (1999) 53-57,
Nucl. Phys. B620 (2002) 407; Journ. Phys. A: Math. Gen. 35 (2002) 9081-9104 hep-th/0102209 hep-th/0110261 hep-th/0301088 and in preparation